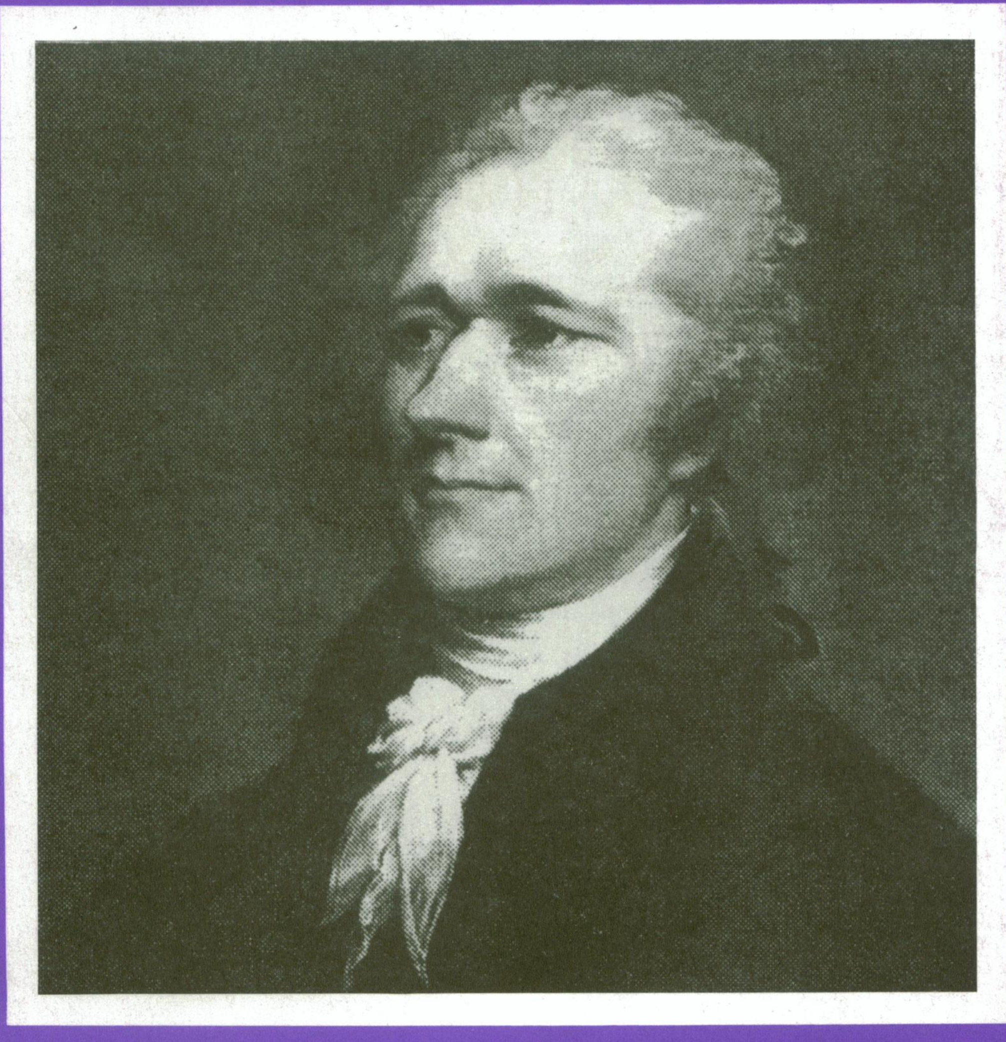
Vol. 65 No. 1, February 1992

MATHEMATICS MAGAZINE



- A Geometric View of Some Apportionment Paradoxes
- Scheduling a Bridge Club
- Rubik's Tesseract

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AUTHOR

Brent A. Bradberry came rather late in life to discover the beauty of mathematics. After 25 years in the U.S. Navy, he came ashore to Washington State University, where he received his Ph.D. under Jack Robertson in 1987. The present article was inspired by a colloquium lecture given by Michael Balinski, and by mathematics curricula using the apportionment problem. Bradberry is now an associate professor of mathematics at Lewis-Clark State College, Lewiston, ID. He is particularly interested in the mathematical preparation of teachers. When not teaching or doing mathematics, he can often be found fishing or hiking.

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ARTICLES

A Geometric View of Some Apportionment Paradoxes

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Introduction

The problem of apportionment is stated in Article I, Section 2 of the U.S. Constitution: "Representatives and direct taxes shall be apportioned among the several states which may be included in this Union, according to their representative numbers.... The number of Representatives shall not exceed one for every thirty thousand but each state shall have at least one representative."

Numerous methods of apportionment have been devised during the past two centuries: None has met with universal approval. Some of the more important methods were devised and championed by outstanding statesmen: Thomas Jefferson, Alexander Hamilton, John Quincy Adams, Daniel Webster. Other methods have been proposed by scholars such as James Dean (Professor of Astronomy and Mathematics, University of Vermont), Joseph A. Hill (Bureau of the Census), and Edward V. Huntington (Professor of Mechanics and Mathematics, Harvard). All these methods, which are analyzed below, can be placed into two categories. In one category is Hamilton's method; in the other (that of Divisor Methods) are all the rest. The methods of Hamilton, Jefferson, and Webster have all been, at various times, the law of the land; the method of Huntington and Hill was passed into law in 1941 and remains the legal method for apportioning the U.S. House of Representatives. The general rule for apportionment methods seems to have been that a method is used until its peculiarities offend enough politicians to warrant its replacement by another method. Balinski and Young [1] provide a fascinating study of the history of the apportionment problem; this paper uses elementary methods to examine and extend some of their results in the context of a geometric model.

Given s states with h seats to apportion, and suppose state s_i has population p_i. The quota is defined as

$$q_i = \frac{p_i}{\Sigma p_i} h,$$

and we may further define the ith state's fair share as the median of (f_i, q_i, c_i), where f_i is the "floor" or minimum number of seats to be given state i, and c_i is the "ceiling" or maximum. According to the U.S. Constitution (quoted above), $f_i = 1$ and c_i is expressed as one seat for each 30,000 people, for each i. For the sake of simplicity this paper will consider the floor to be 0 and the ceiling h.

Balinski and Young have proposed five axioms:

AXIOM I (Population Monotonicity). *No state that gains population gives up a seat to one that loses population.*

AXIOM II (Absence of Bias). *On the average, over time, each state receives its fair share.*

AXIOM III (House Monotonicity). *As the total number of states increases, with populations fixed, no state loses a seat.*

AXIOM IV (Fair Share). *No state's representation deviates from its fair share by one whole seat or more.*

AXIOM V (Near Fair Share). *No transfer of a seat from one state to another brings both nearer to their fair shares.*

The most important result in the history of the apportionment problem is the recent (1982) impossibility theorem of Balinski and Young: *There is no method of apportionment that satisfies axioms I through V* (specifically, no divisor method stays within fair share and a method is population monotone if and only if it is a divisor method), [2] and [3]. This paper will construct a geometric model of apportionment, examine the historic paradoxes resulting from the use of Hamilton's method, and describe a paradox not previously mentioned in the literature.

Geometric Representation of Apportionment

A geometric model of apportionment can be produced by considering the points $x \in E^s, x = (x_1, \ldots, x_s)$, where each $x_i \geqslant 0$ and $\Sigma_{i=1}^{s} x_i = h$ (s is the number of states and h is the house size—i.e. the number of seats to be apportioned); the problem of apportionment then becomes the problem of partitioning the resulting simplex according to agreed upon rules. Further definitions are:

a) *Apportionments* are the lattice points $a = (a_1, \ldots, a_s)$ where each a_i is a nonnegative integer and $\Sigma_{i=1}^{s} a_i = h$.

b) The *populations* $p = (p_1, \ldots, p_s)$ are normalized and represented by the *quotas* $q = (q_1, \ldots, q_s)$ where $q_i = (p_i / \Sigma p_i)h$.

c) An *apportionment region* R_a is the set of populations q for which the apportionment is a, by whatever apportionment rule is in use.

d) An *apportionment diagram* is an apportionment simplex along with all apportionment regions and boundaries.

FIGURE 1 illustrates an apportionment diagram for the allocation of two seats to three states, by a method that evidently works in favor of state 3; the apportionment regions $R_{(0,0,2)}$, $R_{(1,0,1)}$, and $R_{(0,1,1)}$ in which state 3 receives one or both seats comprise most of the area of the apportionment diagram triangle. This type of apportionment will not be considered further, since it is not symmetric. We will insist, *a priori*, that any apportionment must be:

a) *Homogeneous*. The apportionment of s states for any population p must be the same as the apportionment for λp for any $\lambda > 0$. This behavior is ensured by the use of the quota q.

b) *Symmetric*. Renumbering the states will not change the apportionment diagram.

c) *True*. Whenever an apportionment a is exactly proportional to the population p, then a must be the unique apportionment for p when $h = \Sigma_{i=1}^{s} a_i$. For example, if $p = (1000, 2000, 3000)$ then six seats must apportion as $(1, 2, 3)$ and in no other way.

Finally, let $a = (a_1, \ldots, a_s)$ be an apportionment; then a^* is a *neighboring apportionment* if for exactly one j, $0 \leqslant j \leqslant s$, $a_j = a_j^* + 1$ and for exactly one k, $0 \leqslant k \leqslant s$, $a_k + 1 = a_k^*$. Thus, for example, when $h = s = 4$, the apportionment $(1,1,1,1)$ has 12 neighbors, of which $(2,0,1,1)$, $(0,1,1,2)$ and $(0,2,1,1)$ are examples. We shall say that apportionment a is *interior* if $a_i > 0$ for each $i = 1, \ldots, s$. (Note: The geometric effect of a floor > 0 is to disallow any apportionments that are not interior.)

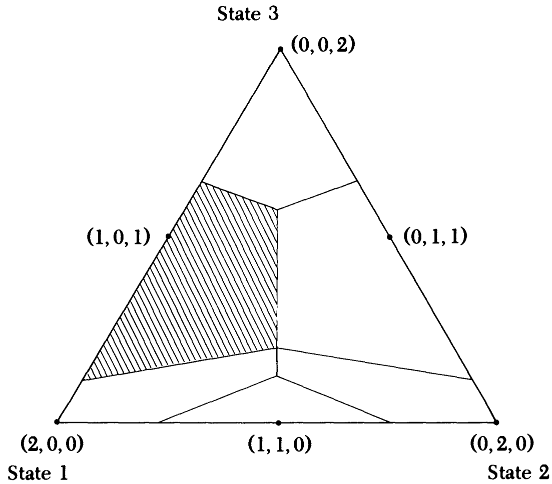

FIGURE 1
An apportionment diagram for $s = 3$, $h = 2$. The shaded area is $R_{(1,0,1)}$, in which states 1 and 3 each receive one seat and state 2 receives none.

Hamilton's Method

Alexander Hamilton's method of apportionment was the first to be adopted by Congress (1792) but was not then used. (George Washington exercised the first presidential veto to stop the bill adopting Hamilton's method, after being persuaded by Thomas Jefferson that he, Jefferson, had a better method. Jefferson's method was then adopted and subsequently abandoned—Hamilton's method was readopted two generations later under the name of representative Samuel F. Vinton, Ohio.) Hamilton's apportionment is accomplished by:

> "Choose the size of the house to be apportioned. Find the quotas and give to each state the whole number contained in its quota. Assign any seats which are as yet unapportioned to those states having the largest fractions or remainders" [2]. That is
>
> a) Form quotas
>
> $$q_i = \frac{p_i}{\Sigma p_i} h,$$
>
> b) Assign to each state its lower quota $[q_i]$; i.e., the integer part of q_i.
>
> c) Assign any remaining seats to the states in order of remainder, that is, the state with the greatest remainder receives the first remaining seat, and so on. The ith state's remainder is $q_i - [q_i]$.

Hamilton's method is simple, straightforward, and clearly within fair share (quota). However, there are three "paradoxes" that affect this method and make it unacceptable:

a) *The Alabama Paradox*. A state may lose representation when the house size increases, even though the number of states and their populations remain unchanged.

b) *The Population Paradox*. Given a fixed house size and a fixed number of states, a given state may lose representation to a second state even though the first state's population is growing at a faster rate.

c) *The New States Paradox*. If a new state enters, bringing in its complement of new seats (that is, the number it should receive under the apportionment method in use), a given state may lose representation to another even though there is no change in either of their populations.

The causes of these paradoxes can be seen by examining the geometry of Hamilton's method. For example, given three states and h seats, the population $q=(q_1,q_2,q_3)$ apportions to $a=(a_1,a_2,a_3)$ if either each $q_i=a_i$ or if any one of the following six conditions hold:

lower quota is	*and*
1) (a_1,a_2-1,a_3)	$q_2-(a_2-1)>\max(q_1-a_1,q_3-a_3)$
2) (a_1-1,a_2-1,a_3)	$q_3-a_3<\min\{q_1-(a_1-1),q_2-(a_2-1)\}$
3) (a_1-1,a_2,a_3)	$q_1-(a_1-1)>\max\{q_2-a_2,q_3-a_3\}$
4) (a_1-1,a_2,a_3-1)	$q_2-a_2<\min\{q_1-(a_1-1),q_3-(a_3-1)\}$
5) (a_1,a_2,a_3-1)	$q_3-(a_3-1)>\max\{q_1-a_1,q_2-a_2\}$
6) (a_1,a_2-1,a_3-1)	$q_1-a_1<\min\{q_2-(a_2-1),q_3-(a_3-1)\}$

In FIGURE 2 the dashed triangle indicates the region in which lower quotas are (a_1,a_2-1,a_3); the boundaries of $R_{(a_1,a_2,a_3)}$ within the triangle are the perpendicular bisectors of the line segments joining (a_1,a_2,a_3) with (a_1,a_2-1,a_3+1) and (a_1+1,a_2-1,a_3), corresponding to the inequalities $q_2-(a_2-1)>q_3-a_3$ and $q_2-(a_2-1)>q_1-a_1$, respectively. Similarly, the dotted triangle represents the region in which lower quotas are (a_1,a_2,a_3-1). (The construction is similar for $s>3$, but increasingly cumbersome as s increases; for $s=4$ there are 12 neighboring apportionments to consider.) The apportionment region R_a is, thus, the region formed by bisecting the line segment joining a to each of its neighbors, by a

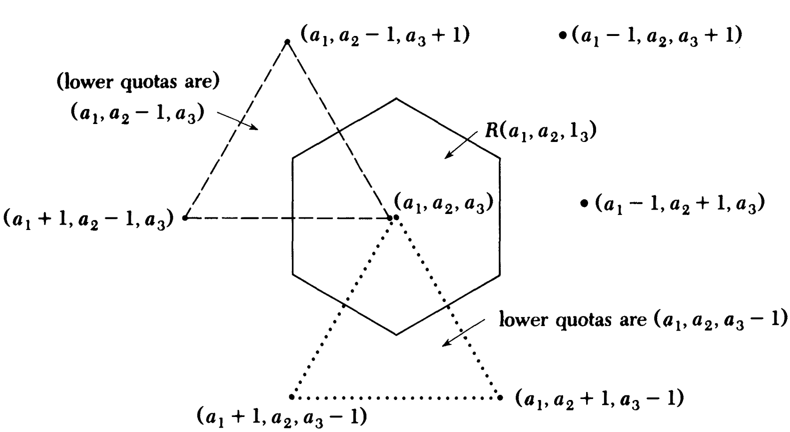

FIGURE 2

hyperplane normal to that line segment. For $s = 3$ the apportionment regions (for interior apportionments) are regular hexagons; for $s = 4$ the interior apportionment regions are regular dodecahedrons. FIGURE 3 illustrates the apportionment regions for Hamilton's method, $s = 3, h = 5$. With the following definition, these results can be summarized in a theorem.

Definition. Let Ψ be a set of discrete points in E^n and let $x = (x_1, \dots, x_n) \in \Psi$. The *Dirichlet domain* of x is the set of points p such that $\rho(p, x) < \rho(p, y)$ for any $y \in \Psi$, $y \neq x$, where ρ is the Euclidean distance [9].

THEOREM 1. *The apportionment region for any apportionment a in a Hamilton apportionment diagram is the Dirichlet domain of a.*

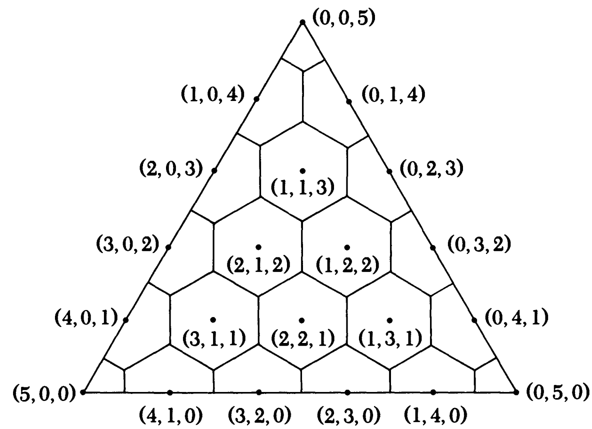

FIGURE 3
Apportionment diagram for Hamilton's method, $s = 3$, $h = 5$.

The Alabama Paradox

Suppose there are three "states" with populations 45, 43, and 12. With three seats to apportion, $q_1 = (45/100) \times 3 = 1.35$, and similarly for q_2 and q_3, so $q = (1.35, 1.29, 0.36)$. The lower quotas are $(1, 1, 0)$ and state 3 has the greatest remainder, so the apportionment is to $(1, 1, 1)$. Now increase the house size to $h = 4$, and we have $q' = (1.80, 1.72, 0.48)$. Lower quotas are $(1, 1, 0)$ but state 3 now has the smallest remainder, hence the apportionment is to $(2, 2, 0)$ and state 3 has lost a seat while the house size increased from three to four.

In general, state i is susceptible to losing a seat due to the Alabama paradox with an increase of seats from h to $h + 1$ whenever q lies in the Dirichlet domain of $a = (\dots, a_i, \dots)$ in the apportionment diagram for h seats, while $q' = (h + 1/h)q$ lies in the Dirichlet domain of $a' = (\dots, a_i - 1, \dots)$ in the apportionment diagram corresponding to $h + 1$ seats. This phenomenon can be illustrated by overlaying the apportionment diagrams for h and $h + 1$; FIGURE 4 overlays the Hamilton apportionment diagram for $s = 3, h = 4$ with that for $s = 3, h = 5$. The shaded areas represent the Alabama paradox-susceptible populations.

In the special case of three states, the Alabama paradox-susceptible regions are parallelograms whose bounding lines are $q_2 = q_3 + n, q_1 = q_3 + m$; $1 \leqslant m, n \leqslant h - 2$.

It is straightforward, although tedious, to calculate the areas of these regions. Then (if we assume that populations are uniformly distributed over the apportionment diagram) we can assert that the probability of occurrence of the Alabama paradox is given by the ratio of the areas of the susceptible regions to that of the apportionment diagram. This leads to:

THEOREM 2. *Under the assumption that populations are uniformly distributed over the apportionment diagram, in the apportionment by Hamilton's method of h seats among three states, the Alabama paradox has probability of occurrence*

$$(1/12)[(h-1)(h-2)/h(h+1)].$$

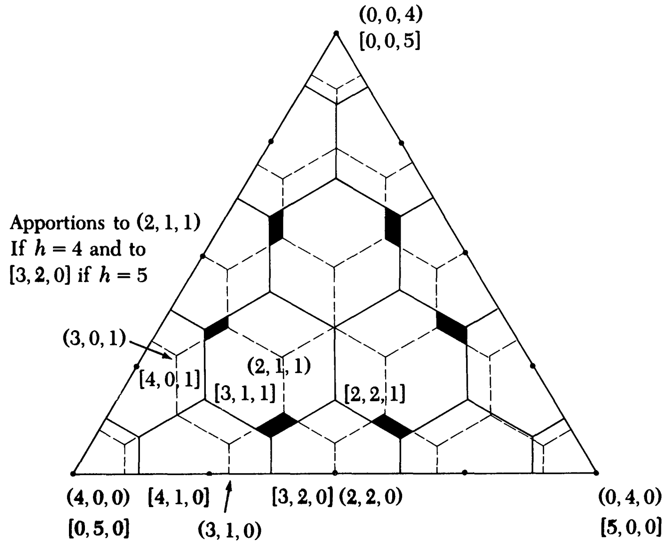

FIGURE 4
Hamilton apportionment diagram for $s = 3$, $h = 5$ (dotted lines and apportionments in square brackets) overlaid on Hamilton apportionment diagram for $s = 3$, $h = 4$ (solid lines and round brackets), with a few apportionments labeled. Populations in the shaded regions are susceptible to the Alabama Paradox.

The Population Paradox

Fix house size h but let populations increase; state i may lose a seat to state j even if state i's population is growing at a faster rate than state j's. If the initial population is p and after some time the population is p', the statement "state i's population is growing faster than state j's" means that

$$\frac{p'_i}{p'_j} > \frac{p_i}{p_j}$$

or, equivalently,

$$\frac{q'_i}{q'_j} > \frac{q_i}{q_j}.$$

Thus, a population increase can cause state i to lose a seat to state j only if simultaneously q lies in the Dirichlet domain of $a = (\dots, a_i, \dots, a_j, \dots)$ while q' lies in that of $a' = (\dots, a_i - 1, \dots, a_j + 1, \dots)$, with the inequality above satisfied. FIGURE 5 illustrates this phenomenon for $s = 3, h = 2$. The vertices and areas of Population paradox-susceptible regions can be calculated in a manner similar to that used for the Alabama paradox:

THEOREM 3. *In the apportionment of h seats among three states using Hamilton's method, those parts of the apportionment diagram in which state* 3 *may lose a seat to state* 2 *through the Population paradox are those triangles whose vertices are*

$$\left(h - s - t + \frac{1}{3}, \frac{3s+1}{3}, \frac{3t-2}{3}\right)$$

$$\left(h - s - t - \frac{1}{3}, \frac{3s+2}{3}, \frac{3t-1}{3}\right)$$

and

$$\left(h + \frac{(s+2t-1)(s-t+1)}{s+2t}, \frac{(s+2t-1)(3s+2)}{3(s+2t)}, \frac{(s+2t-1)(3t-1)}{3(s+2t)}\right)$$

for $1 \leqslant t \leqslant s,\ s + t \leqslant h$.

Since any state may be chosen to lose a seat to any other state, six-fold symmetry can be used to produce all the Population paradox-susceptible areas. FIGURE 6 illustrates the susceptible areas for $s = 3, h = 3$. As a numerical example, suppose $s = 3, h = 3$, and the populations at some time t_1 are 420, 455, and 125, respectively, while at a later time t_2 the populations are 430, 520, and 150. All states have experienced growth, and the fastest-growing state is s_3. However, $q_{t_1} = (1.26, 1.36, .38)$, which results in a Hamiltonian apportionment to $(1, 1, 1)$, while $q_{t_2} = (1.17, 1.42, .41)$, which apportions to $(1, 2, 0)$. State 3 loses its seat to the more slowly growing state 2.

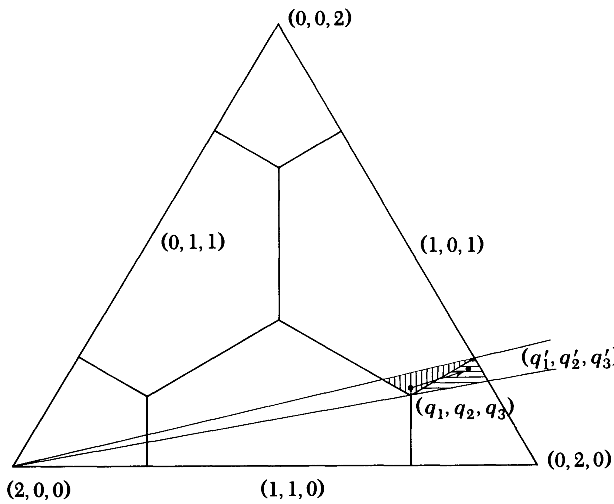

FIGURE 5

The Population Paradox. A population change from (q_1, q_2, q_3) to (q'_1, q'_2, q'_3) causes state 3 to lose a seat even though $q'_3/q_3 > q'_i/q_i$ for $i = 1, 2$.

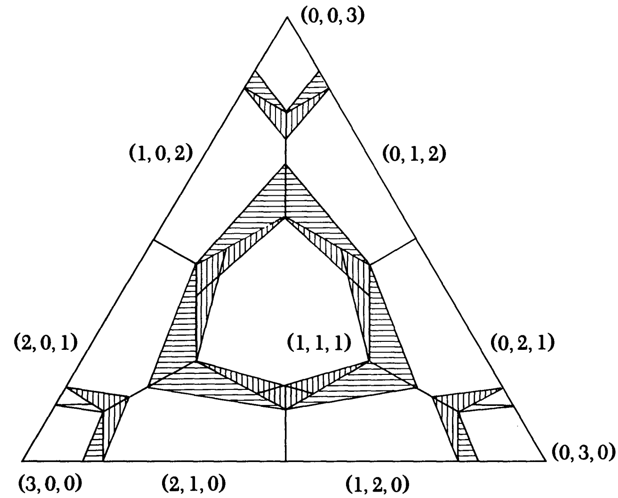

FIGURE 6

The Population Paradox. A population change from a vertically hatched region to a horizontally hatched one (or vice versa) may cause a state to lose a seat to a state with a lower population growth rate.

The New States Paradox

Suppose a new state enters and brings with it the proper number of new seats (i.e. its fair share); under what conditions may state i lose a seat to state j, even though there is no change in the ith and jth populations? In order to study this situation, consider an apportionment of h seats among three states and ask "If a population $q = (q_1, q_2, q_3)$ apportions h seats to $a = (a_1, a_2, a_3)$, under what conditions will the population $q' = (q_1, q_2)$ apportion $h - a_3$ seats to $a' = (a_1 + 1, a_2 - 1)$?" First note that for $s = 2$ (a two-state apportionment problem), the apportionments can be modeled on a number line as:

$(h, 0)$ —— $(h-1, 1)$ —— $\ldots\ldots$ $(h-2, 2)$ —— $(0, h)$

The Dirichlet domains of the apportionments are the intervals $(r - \frac{1}{2}, r + \frac{1}{2})$ for integers r.

Now let

$$q_i = \frac{p_i}{p_1 + p_2 + p_3} h \quad \text{and} \quad q_i' = \frac{p_i}{p_1 + p_2}(h - a_3).$$

Then the population (p_1, p_2) apportions to $(a_1 + 1, a_2 - 1)$ if

$$a_1 + \frac{1}{2} < q_1' < a_1 + \frac{3}{2},$$

which simplifies to

$$\frac{2a_1 + 1}{2a_2 - 1} < \frac{q_1}{q_2} < \frac{2a_1 + 3}{2a_2 - 3}.$$

This inequality represents a wedge originating at the vertex $(0,0,h)$. Thus, the New States paradox-susceptible area is that area, if any, included in the intersection of the wedge with R_a. FIGURE 7 illustrates this phenomenon for a population $q=(q_1,q_2,q_3)$ that apportions three seats as $(0,2,1)$, while the population $q'=(q_1,q_2)$ apportions two seats as $(1,1)$. FIGURE 8 illustrates all New States paradox-susceptible areas for $s=3, h=4$. We can calculate the vertices and areas of these susceptible areas in a manner similar to that used in the prior cases.

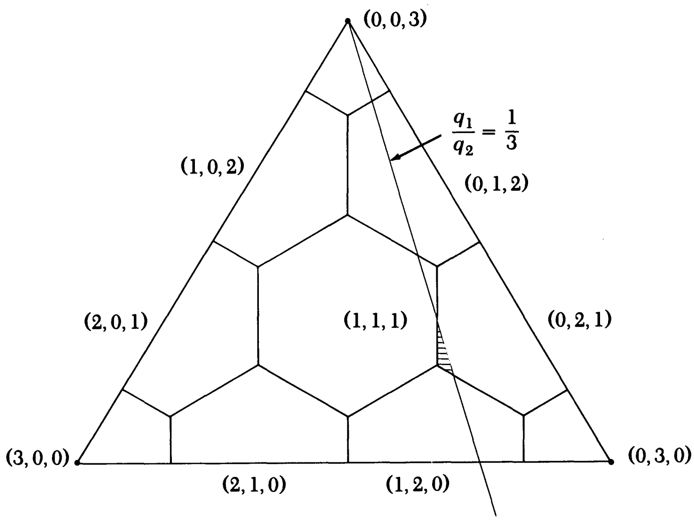

FIGURE 7

The New States Paradox. The cross-hatched region represents populations that would cause state 1 to lose a seat (to state 2) if state 3 joins the union, bringing with it a new seat.

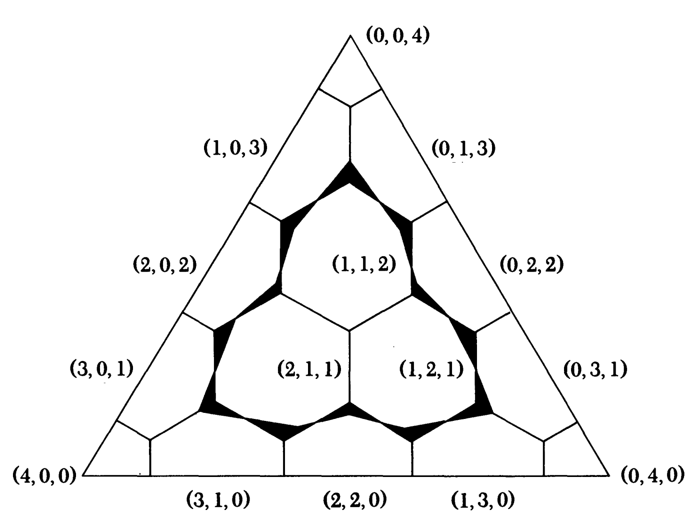

FIGURE 8

The New States Paradox. Populations represented by shaded regions are susceptible to the New States Paradox.

THEOREM 4. *Using Hamilton's method to apportion h seats among three states, state 1 is susceptible to losing a seat to state 2 by the New States paradox in the regions which are the intersections of $R_{(r,s,t)}$ with the wedge*

$$\left(\frac{2r+1}{2s-1}<\frac{q_1}{q_2}<\frac{2r+3}{2s-3}\right)$$

for $s \geqslant 2, r \leqslant s-2, r+s \leqslant h-1$. Six-fold symmetry can be employed to describe all New States paradox-susceptible regions.

As a numerical example of this paradox, consider the Hamiltonian apportionment of four seats to two states whose populations are 623 and 377. Now suppose a new state (population 200) joins the union and the house size is increased to five. In the first case $q=(2.49, 1.51)$ so states 1 and 2 each receive two seats. After the addition of state three, $q=(2.60, 1.57, 0.83)$ and state 2 has lost a seat to state 1 since the apportionment is now $(3, 1, 1)$.

Divisor Methods

The susceptibility of Hamilton's method to the three paradoxes described above finally caused it to be dropped from consideration as a method of apportioning the House of Representatives; only the divisor methods remain. Divisor methods are all based on the fact that, no matter what rounding rule is chosen, when quotas are rounded the results frequently do not sum to the required house size h. To rectify this situation, divisor methods first divide populations (or, equivalently, quotas) by a common divisor that is chosen so that the resulting quotients, when rounded, sum to h. That is, in any divisor method, $q \in R_a$ if, for some common divisor $x>0, [q_i/x]=a_i$ for each $i=1,\dots,s$ where [ ] denotes the rounding rule. An equivalent way of expressing the above is $q \in R_a$ if, for some $x>0, b_{a_i}<q_i/x<b_{a_i+1}$; $i,\dots,s$; where the b_{a_i} are the specified rounding points (these rounding points can conveniently be visualized as flags placed along a number line [2]). For convenience we adopt the convention that $b_0=0$ and $b_i \leqslant b_j$ whenever $i \leqslant j$.

The system of inequalities

$$b_{a_i}<\frac{q_i}{x}<b_{a_i}+1$$

represent wedges originating from the vertices of the apportionment diagram. Some manipulation results in:

THEOREM 5. *For a divisor method of apportionment with rounding points $0=b_0 \leqslant b_1 \leqslant \cdots \leqslant b_h$ where h seats are to be apportioned among three states, the apportionment region $R_{(r,s,t)}$ is the intersection of the wedges*

$$\frac{b_r}{b_{s+1}}<\frac{q_1}{q_2}<\frac{b_{r+1}}{b_s}$$

$$\frac{b_r}{b_{t+1}}<\frac{q_1}{q_3}<\frac{b_{r+1}}{b_t}$$

$$\frac{b_s}{b_{t+1}}<\frac{q_2}{q_3}<\frac{b_{s+1}}{b_t}.$$

FIGURE 9 illustrates a typical divisor apportionment region.

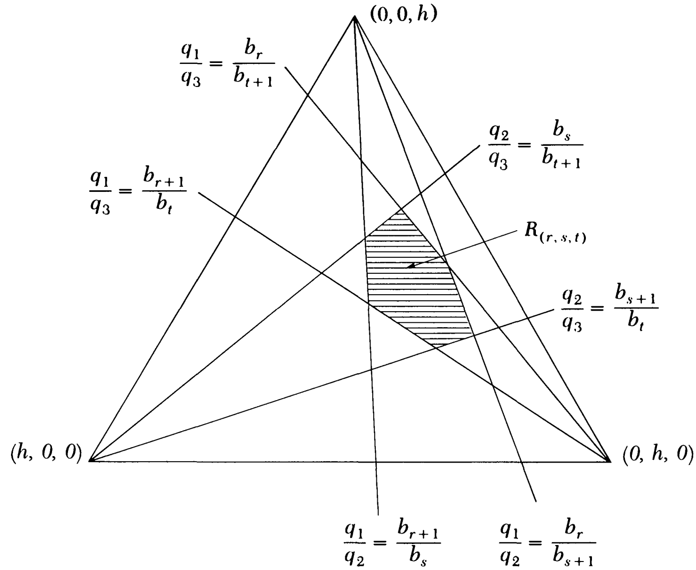

FIGURE 9

A typical divisor method apportionment region and its boundaries.

Jefferson's method can be stated: "Choose the size of the house to be apportioned. Find a divisor x so that the whole numbers contained in the quotients of the states sum to the required total. Give to each state its whole number" [**2**]. That is, the rounding rule is to round all quotients down—in terms of flags, $b_i = i$ for $i = 0, 1, \ldots, h$. FIGURE 10 illustrates Jefferson's method for $s = 3, h = 5$. Note the hatched regions; they denote violations of fair share. For instance, at the top of the figure populations in the hatched region are apportioned to $(0, 0, 5)$ even though $q_3 < 4$. Violations of quota, while not possible in Hamilton's method, occur in all divisor methods [**3**].

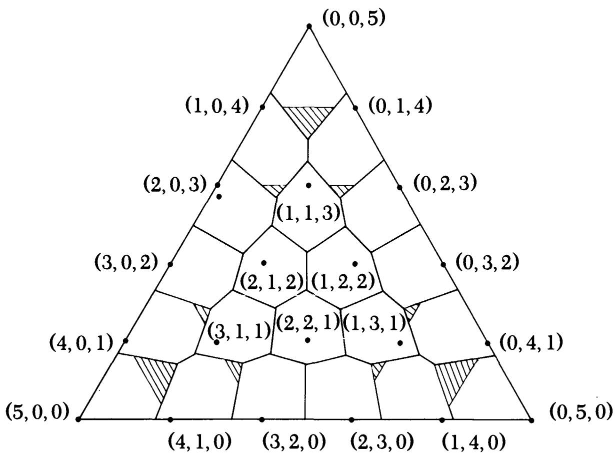

FIGURE 10

Apportionment diagram for Jefferson's method, $s = 3$, $h = 5$. Populations in the hatched regions apportion in violation of quota.

Adams' method is the mirror image of Jefferson's: "Choose the size of the house to be apportioned. Find a divisor x so that the smallest whole numbers containing the quotients of the states sum to the required total. Give to each state its whole number" [2]. That is quotients are rounded up, so $b_i = i - 1$ for $i = 1, 2, \ldots, h$.

In Dean's method, the ith state receives a_i seats where p_i/a_i is as close as possible to the common divisor x. That is, for all i we have

$$\frac{p_i}{a_i} - x \leqslant x - \frac{p_i}{a_i + 1} \quad \text{and} \quad x - \frac{p_i}{a_i} \leqslant \frac{p_i}{a_i - 1} - x,$$

which simplifies to

$$\frac{\left(a_i + \frac{1}{2}\right)}{a_i(a_i + 1)} p_i \leqslant x \leqslant \frac{\left(a_i - \frac{1}{2}\right)}{a_i(a_i - 1)} p_i \quad (\text{for all } i = 1, \ldots, h).$$

Now set

$$d(a+1) = \frac{a(a+1)}{a + \frac{1}{2}}$$

and we have

$$\max_i \left(\frac{p_i}{d(a_i + 1)} \right) \leqslant x \leqslant \min_j \left(\frac{p_j}{d(a_j)} \right).$$

Thus the rounding points for Dean's method are $b_{i+1} = i(i+1)/(i + \frac{1}{2})$, $i = 0, 1, \ldots, h - 1$; that is, the harmonic means of the consecutive integers [2].

In Webster's method, the rounding points are the "natural" ones,

$$b_i = \frac{2i - 1}{2}, \qquad i = 1, 2, \ldots, h.$$

The Huntington-Hill method can be stated as: "Choose the size of the house to be apportioned. Give to each state a number of seats so that no transfer of any one seat can reduce the percentage difference in representation between those states [2]." That is, supposing that state i is favored over state j, or

$$\frac{a_i}{p_i} > \frac{a_j}{p_j};$$

no transfer of seats will be made if

$$\frac{\dfrac{a_i}{p_i} - \dfrac{a_j}{p_j}}{\dfrac{a_j}{p_j}} < \frac{\dfrac{a_j + 1}{p_j} - \dfrac{a_i - 1}{p_i}}{\dfrac{a_i - 1}{p_i}}$$

for all i, j. This expression simplifies to

$$\max_j \left(\frac{p_j}{\sqrt{a_j(a_j + 1)}} \right) < \min_i \left(\frac{p_i}{\sqrt{a_i(a_i - 1)}} \right).$$

Thus the rounding rule for Huntington-Hill is $d(a) = \sqrt{a(a-1)}$, or the geometric means of consecutive integers [2]. In terms of flag placement, $b_i = \sqrt{i(i-1)}$, $i = 1, \dots, h$.

FIGURES 10–12 illustrate apportionment diagrams for $s = 3, h = 5$ for the methods of Jefferson, Adams, and Webster, respectively. Diagrams for the Dean and Huntington-Hill methods look similar to Webster's. Compare FIGURES 10–12 with FIGURE 3 (Hamilton's method for $s = 3, h = 5$).

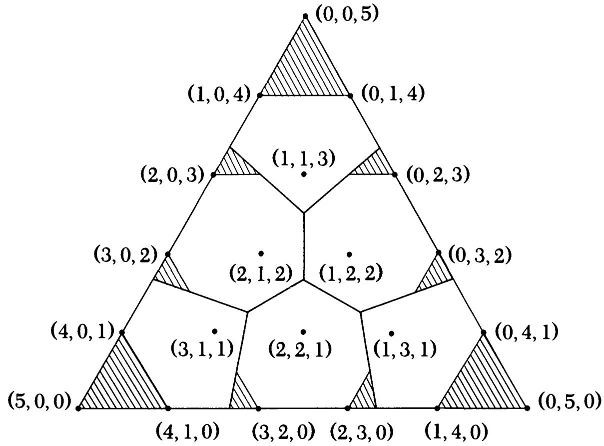

FIGURE 11
Apportionment diagram for Adams' method, $s = 3$, $h = 5$. Populations in the hatched regions apportion in violation of quota.

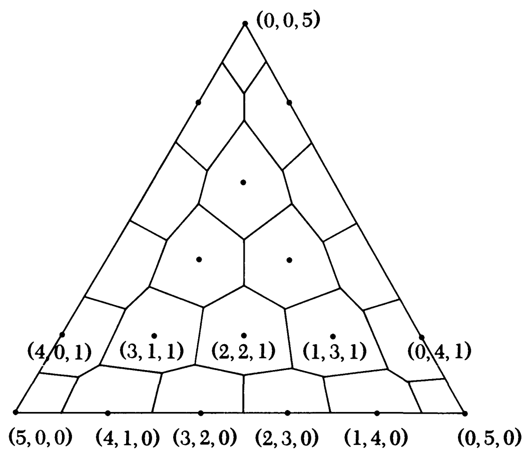

FIGURE 12
Apportionment diagram for Webster's method, $s = 3$, $h = 5$. Webster's method does not violate quota for $s = 3$, $h = 5$.

The Migration Paradox

There is a "paradox" not previously mentioned in the literature that the author has chosen to call the Migration paradox (it could as well be called the Innocent Bystander paradox, as will be seen). This paradox affects both Hamilton's method and the divisor methods:

The Migration Paradox. Let total population, house size, the number of states, and the population of a given state, say the ith state, all be fixed. It is possible that state i can lose representation if there is a population shift (migration) between two other states.

For example, suppose we are apportioning three seats among three states by Webster's method and $p = (1000, 400, 1600)$. The divisor $x = 1000$ produces quotients 1, 0.4, and 1.6 respectively, resulting in the apportionment $a = (1, 0, 2)$. Now suppose 250 people move from state 1 to state 2, resulting in $p' = (750, 650, 1600)$. The number 1000 can't be used as a divisor in this case, since four seats would be required to produce an apportionment. A larger divisor, say 1080, produces quotients $0.69, 0.60, 1.48$, so the apportionment is $(1, 1, 1)$ and state 3 has lost a seat although its population (and quota) have not changed. FIGURE 13 illustrates this phenomenon—note that state 3 could, apparently, have a slight population increase and still lose a seat under this paradox; furthermore, a larger migration (if, say, 600 people moved from state 1 to state 2) would have left state 3 with its two seats intact.

The Question of Bias

With Hamilton's method out of contention, it is now natural to seek a means of choosing between divisor methods. As early as 1832, it was well recognized by Congress that some apportionment methods tended to favor (i.e., were biased toward) smaller states while other methods favored larger states. Jefferson's method favored larger states, while Adams' method favored smaller states. The methods of Webster, Dean, and Hill (herein called the Huntington-Hill method) are all attempts at balance, or compromise. (Compare the relative sizes of various apportionment regions in, say, FIGURES 10 and 11.) Indeed, the chief virtue claimed for the Huntington-Hill method, which is currently in use, is that it is somehow midway between the methods of Jefferson and Adams [2]. Huntington was able to convince many, including John von Neumann, of the superiority of his method [3].

Walter Willcox, a Cornell statistician and, among other accomplishments, president of the American Economic Association, disagreed with Huntington. Willcox published articles on apportionment in 1910, 1916, and 1952. He believed that Webster's method was the least biased of the historical methods and, in fact, in 1916 he stated "the use of [the Huntington-Hill method] inevitably favors the small state." [2] Unfortunately, neither Huntington nor Willcox gave precise definitions of what was meant by bias, nor did either of them present cogent mathematical arguments to support their contentions.

Balinski and Young were the first to define bias precisely and they concluded that Webster's method is the best choice. Readers interested in further study of the apportionment problem are encouraged to read Balinski and Young's excellent book [2]. Although they were the first to offer proof that Webster's method is the best of the historical methods used to apportion the U.S. House of Representatives, they were anticipated, at least intuitively, by Representative John A. Anderson of Kansas. They cite the following statement by Mr. Anderson on the floor of the house in 1882:

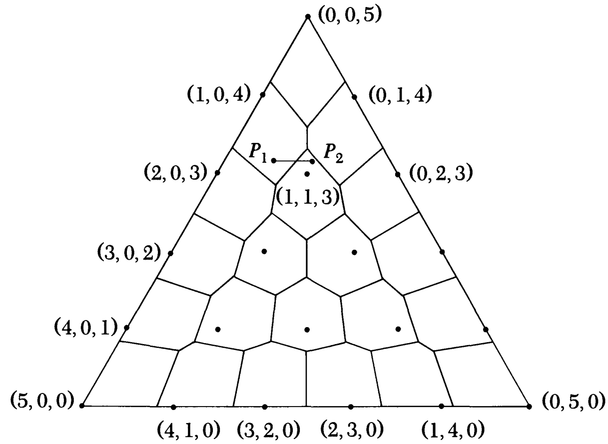

FIGURE 13

The Migration paradox, shown in Jefferson's method, $s = 3$, $h = 5$. A shift of population from, e.g., P_1 to P_2 (produced by a migration from state 1 to state 2 that leaves state 3's population unchanged) causes a change in apportionment from (1,0,4) to (1,1,3), so state 3 has lost a seat.

"Since the world began there has been but one way to proportioning numbers, namely, by using a common divisor, by running the "remainders" into decimals, by taking fractions above .5, and dropping those below .5; nor can there be any other method. This process is purely arithmetical... If a hundred men were being torn limb from limb, or a thousand babes were being crushed, this process would have no more feeling in the matter than would an iceberg; because the science of mathematics has no more bowels of mercy than has a cast-iron dog."

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NOTES

Scheduling a Bridge Club (A Case Study in Discrete Optimization)

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Introduction Interesting mathematics problems arise from a wide variety of sources in everyday life. This paper explores a scheduling problem that was presented to one of the authors by a local bridge club. Although the problem appears simple on the surface, analysis uncovers a complexity that is often present in simply stated discrete optimization problems, and illustrates pitfalls that can arise from naively applying brute force techniques on the computer. The paper first defines the problem and explores the meaning of an optimal solution. Next an analytical solution is sought based on the classification of the problem, and finally the paper considers four increasingly sophisticated techniques of discrete optimization.

Historically, optimization problems have arisen in a variety of applications including electrical engineering, operations research, computer science, and communication. Although a variety of techniques for solving linear and non-linear optimization problems with continuous variables has been well known for 25 years [**2, 4, 15**], it is only recently that progress has been made in solving optimization problems involving discrete variables [**9, 10**]. This paper examines a variety of these discrete techniques in the context of solving one specific scheduling problem.

The problem A bridge club consists of 12 couples. At each club meeting, the club is divided into 3 groups of 4 couples. Each of the 4 couples then competes against the remaining 3 couples in its group, which requires 6 games per group, 18 for the meeting. These meetings occur 8 times a year, resulting in a total of 144 games. The club has requested a schedule for the year with the following property: The number of times any couple competes against any opposing couple is the same for all couples. If no such schedule exists, then the club requests a schedule where the number of times any couple plays any other couple is most nearly the same for all couples.

Optimal solutions Since each couple competes 8 times against 3 opposing couples, there are 24 potential opposing couples (teams). Since a couple does not play itself, there are only 11 different possible opponents. Hence an "optimal" schedule would have each couple playing 9 other couples twice and 2 other couples three times. However, there is no guarantee that such an "optimal" schedule exists.

The method we shall use to measure optimality makes use of a competition matrix C. The i, jth element of C (denoted C_{ij}) is the number of times team i competes against team j. This matrix is symmetric ($C_{ij} = C_{ji}$), with 0 on the diagonal, so only the 66 elements below (or above) the main diagonal need to be examined. Another feature of the competition matrix is that the sum of all elements of C is a constant (288) regardless of the schedule used, because this sum is twice the number of games (144) played. If an "optimal" schedule exists, each row of C would consist of nine 2's and two 3's (with 0's on the diagonal). Thus above the diagonal there would be fifty-four 2's and twelve 3's.

The cost of a particular schedule is computed by summing the squares of the difference between the average number of times two teams should meet, namely 2, and each of the entires above the main diagonal of the competition matrix. That is,

$$\text{Cost} = \sum_{i<j}^{12} (C_{ij} - 2)^2. \tag{1}$$

In this way, the cost of a schedule is decreased when the competition matrix has more entries that are closer in value to 2. The minimum possible cost is thus seen to be 12. (It should be noted that all cost functions of the form $\Sigma[C_{ij}]^n$ for $n > 1$ will produce higher costs for entries that are unbalanced. This can be seen by using the method of Lagrange multipliers to find that the minimum of $\Sigma[C_{ij}]^n$ subject to the constraint $\Sigma[C_{ij}] = 288$ occurs when all C_{ij} are equal. Experience has shown that all cost functions of this form will produce similar results.)

Classification (resolvable partially balanced incomplete block design) The scheduling problem can be classified as one in experimental *block design*. This field of mathematics is concerned with the construction and design of experiments for which there are several variables that are all being changed simultaneously. The best experimental design has the property that the effects of varying parameters can be easily factored out. Block design originally was used to design experiments using different types of plants on several plots of land [3]. If all plants of the same type were grown in the same plot, the effect of the type of soil in that particular plot could affect the experiment, and hence the plots and plants should be mixed. The two types of parameters (plots and plants in the above example) are called blocks and varieties. In our scheduling problem a block refers to a group of four teams and a variety refers to a specific team. The number of varieties will be denoted by v and the number of the blocks by b. In our example, $v = 12$ and $b = 24$.

The design is called an *incomplete block design* if the number of blocks and the number of varieties are not equal. For a solution to the scheduling problem, no variety will appear more than once in a block, each block will contain the same number (denoted k) of varieties, and all varieties will appear in the same number (denoted r) of blocks. In our problem $k = 4$ and $r = 8$. By counting the appearance of varieties in the blocks in two different ways, it is evident that

$$bk = vr. \tag{2}$$

In a *balanced block design*, the number of times any pair of varieties appears together is a constant (denoted λ). In a balanced incomplete block design (BIBD) it can be shown [11] that

$$r(k-1) = \lambda(v-1). \tag{3}$$

Our preliminary analysis has shown that the best schedule for our problem will have most teams meeting twice while some will meet three times and hence a balanced block design is not possible. Thus we desire a schedule that is as balanced as possible.

A further property that is required of our schedule is that the blocks be designed in such a way that they can be grouped into 8 sets, where the varieties in the 3 blocks of any of these 8 sets represent exactly the teams 1–12. This property disallows schedules where a team doesn't play at a meeting or some teams play more than once. This property has the technical name of *resolvability*.

Although there are procedures for constructing unbalanced block designs, these procedures are usually restricted to very specific classes of problems. Unfortunately it appears that this scheduling problem doesn't fall into any class of solved problems. A solution could be sought using balanced block design techniques. A balanced block design could then be utilized in one of two ways: either a partial schedule with $\lambda = 2$ or $\lambda = 1$ can be found and then augment it with more blocks (meetings), or an over-full schedule can be found with $\lambda = 3$ (all pairs of teams meet thrice) and then remove some meetings. For any option, the number of teams per block must be 4, and the number of varieties (teams) is 12. If the first option ($\lambda = 2$ or $\lambda = 1$) is chosen then the application of equation (3) yields that $r = 22/3$ or $11/3$, respectively. Since this is not an integer, no such balanced incomplete block design exists. This leaves only the second option ($\lambda = 3$). The application of equations (3) and then (2) yield $b = 33$, $v = 12$, $r = 11$, $k = 4$, and $\lambda = 3$. The existence of such a balanced incomplete block design is guaranteed by [5]. However, a schedule produced by removing 9 of the 33 blocks is not necessarily resolvable.

Method 1 (exhaustive search) The first discrete optimization method considered is to find the optimal solution by an exhaustive search of all possible schedules. To implement this approach, label the groups at each meeting A, B, and C; each group contains 4 couples. The number of possible assignments of the 12 couples to these three groups is given by $C(12,4)C(8,4) = 34{,}650$. The first factor represents the number of ways of choosing four teams to be in group A and the second term indicates how many ways there are of then selecting 4 teams for group B. The remaining 4 teams go into group C. But the labels on the groups are quite arbitrary, and one particular division of the 12 teams into 3 groups of 4 teams each could be labeled in 3! different ways. Thus, dividing by 6 gives us the number of distinct possibilities for each meeting: 5775.

Thus meeting 8 times a year, there are $(5775)^8 = 1.24 \times 10^{30}$ possible schedules, but many of these schedules are essentially equivalent. For example, since the numbering of couples is arbitrary, we may as well assume that for the first meeting, the couples are grouped 1-4 in A, 5-8 in B, and 9-12 in C. Moreover, it won't really matter in what order the remaining 7 meetings are held, as long as all 8 meetings have distinct arrangements. Since one particular arrangement has been chosen for the first meeting, we need only choose 7 of the remaining possibilities. There are thus $C(5774,7) = 4.23 \times 10^{22}$ different schedules. If a computer checked 100 different schedules per second (a reasonable number for a mainframe computer) the search of all possible schedules would require about 1.34×10^{13} years. This is at least 600 times as long as the age of the universe! This is not a very feasible method.

Method 2 (greedy algorithm) The second method considered makes use of a *greedy algorithm*. This method attempts at each successive meeting to minimize the total number of repeated pairings of teams, without changing the schedules of the previous meetings. The method is greedy in the sense that it assumes that the best schedule in the short run will be part of the best schedule in the long run.

More specifically the algorithm begins by choosing an arbitrary schedule for the first evening. It then looks at all remaining unused one-evening schedules and chooses one that when adjoined to the previously chosen schedules, minimizes the total number of repeated pairings between all teams. It repeats this step until all 8 evenings have been scheduled.

For the first meeting any one specific schedule is chosen. For the second meeting we examine all schedules except for the one used at the first meeting (5774 schedules) since repeating a schedule will pair up teams that have been previously paired, and hence will not be optimal. For the third meeting we examine all schedules but the two used in the first and second meetings (5773). Continuing in this manner we examine 5774 + 5773 + 5772 + 5771 + 5770 + 5769 + 5768 = 40,397 total schedules. Examining four possible arrangements every second (a reasonable number on a PC) will require just under three hours to complete the algorithm.

This method was implemented and produced a schedule with four 1's, forty-seven 2's, fourteen 3's and one 4 above the diagonal, or a cost of 22 as compared to the "optimal" cost of 12.

Greedy algorithm results

Schedule
(entries are group label assigned to team)

Team #	Meeting # 1	2	3	4	5	6	7	8
1	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
2	*A*	*A*	*B*	*B*	*B*	*B*	*B*	*B*
3	*A*	*B*	*C*	*C*	*C*	*A*	*B*	*C*
4	*A*	*C*	*C*	*C*	*B*	*C*	*C*	*B*
5	*B*	*B*	*C*	*B*	*A*	*C*	*B*	*A*
6	*B*	*B*	*B*	*A*	*B*	*A*	*C*	*B*
7	*B*	*A*	*B*	*C*	*C*	*C*	*A*	*C*
8	*B*	*C*	*A*	*B*	*C*	*B*	*C*	*C*
9	*C*	*C*	*B*	*B*	*A*	*A*	*A*	*A*
10	*C*	*C*	*A*	*A*	*B*	*C*	*B*	*C*
11	*C*	*A*	*C*	*A*	*C*	*B*	*C*	*A*
12	*C*	*B*	*A*	*C*	*A*	*B*	*A*	*B*

Cost of this schedule is 22.

Competition Matrix
(entries are # of times team plays opponent)

Team #	Opponent # 1	2	3	4	5	6	7	8	9	10	11	12
1	*	2	2	1	2	2	2	1	4	2	3	3
2	2	*	2	3	2	3	2	2	2	2	2	2
3	2	2	*	3	3	2	3	2	1	2	2	2
4	1	3	3	*	2	3	2	2	1	3	2	2
5	2	2	3	2	*	2	2	2	3	2	2	2
6	2	3	2	3	2	*	2	2	2	2	2	2
7	2	2	3	2	2	2	*	3	2	2	2	2
8	1	2	2	2	2	2	3	*	2	3	3	2
9	4	2	1	1	3	2	2	2	*	2	2	3
10	2	2	2	3	2	2	2	3	2	*	2	2
11	3	2	2	2	2	2	2	3	2	2	*	2
12	3	2	2	2	2	2	2	2	3	2	2	*

Approximate Personal Computer Time (3 hours).

Method 3 (branch and bound) Another possible method attempts to examine all relevant schedules by *branch and bound* [13, 16]. In this method, each meeting's schedule is considered to be a node on a tree. The tree's root is connected to 5,775 nodes, one for each of the 5,775 possible schedules for the first meeting. Each node at this first level of the tree would also have 5,775 branches, one for each possible second meeting schedule that assumes the first meeting's schedule of the node that fathered it. In this manner a tree with 8 levels could be formed with all possible final (eight meetings) schedules at the bottom of the tree.

The lowest level of the tree contains the full $(5{,}775)^8$ possible schedules. However, if one could eliminate all branches of the tree below an ith level node, one would eliminate $(5{,}775)^{8-i}$ possible schedules. This is the strategy of branch and bound. One eliminates entire regions of the tree from consideration and searches the rest of the tree. The earlier (closer to the root) the elimination, the fewer possible schedules left to examine. The criterion used to eliminate branches from the tree is based on the characteristics of the optimal schedule. In this example it can be assumed (based on the results of method 2) that in the optimal schedule, the number of times team i meets team j must be between one and four.

This tree can be initially pruned by arbitrarily choosing a schedule for the first meeting. Then there are only $(5{,}775)^7$ possible schedules to search. Unfortunately, this does not have a significant effect on reducing the number of possibilities. The criterion that teams must compete at least once does not eliminate possibilities until going to the 8th level, while the criterion that teams cannot meet more than four times does not eliminate possible schedules until going from the 4th to the 5th level. Since the proportion of schedules that are eliminated by the above criteria is difficult to calculate, the exact number of schedules that need to be searched is not known. However, even if all schedules but one are eliminated beyond the 4th level, there are still $(5775)^4 = 1.11 \times 10^{15}$ possible schedules to search. Again examining 100 schedules per second requires 3.52×10^5 years, so this method is also impractical.

The problem lies in the fact that the tree has very few (only seven) levels (places where a decision, and hence elimination of branches, can take place). The number of decisions that can be made before a final schedule is found can be increased by replacing the node that represents an ith meeting schedule, with an 11 level tree, one level for the placement of each of the 11 teams (team one is always in group A). The tree now has 77 levels, and the optimizing criteria can be used much more frequently.

The tree is produced by first placing team 1 in group A for each of 8 meetings. We then attempt to successively place each of the remaining teams in each of the three groups. If the group is full or the placement violates our optimal criteria, we skip that placement. Skipping a placement is equivalent to removing a branch of the tree and all the nodes that are below it. Once all 12 teams are placed, we again attempt to place each of the 12 teams in all groups for the next meeting. Once all 8 meetings have been chosen, we save the resulting 8-meeting schedule if it is an improvement over the best 8-month schedule we have examined so far. We then move back up the tree to the first node where we can make a legal alternative choice, and proceed as before. Note that we only examine schedules that meet our optimal criteria and that we stop examining any schedule that will lead to an 8-meeting schedule that will not meet our optimal criteria.

Greedy (1-4)/branch and bound (5-8) results

Schedule
(entries are group label assigned to team)

Team #	Meeting # 1	2	3	4	5	6	7	8
1	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
2	*A*	*A*	*B*	*B*	*B*	*B*	*A*	*B*
3	*A*	*B*	*C*	*C*	*A*	*B*	*C*	*C*
4	*A*	*C*	*C*	*C*	*B*	*C*	*B*	*A*
5	*B*	*B*	*C*	*B*	*C*	*C*	*A*	*C*
6	*B*	*B*	*B*	*A*	*B*	*A*	*B*	*A*
7	*B*	*A*	*B*	*C*	*A*	*C*	*C*	*B*
8	*B*	*C*	*A*	*B*	*C*	*B*	*C*	*A*
9	*C*	*C*	*B*	*B*	*A*	*A*	*B*	*C*
10	*C*	*C*	*A*	*A*	*B*	*C*	*C*	*C*
11	*C*	*A*	*C*	*A*	*C*	*B*	*B*	*B*
12	*C*	*B*	*A*	*C*	*C*	*A*	*A*	*B*

Cost of this schedule is 20.

Competition Matrix
(entries are # of times team plays opponent)

Team #	Opponent # 1	2	3	4	5	6	7	8	9	10	11	12
1	*	3	2	2	1	3	2	2	2	2	2	3
2	3	*	2	2	2	2	3	2	2	1	3	2
3	2	2	*	3	3	1	3	2	2	2	2	2
4	2	2	3	*	2	3	2	2	2	3	2	1
5	1	2	3	2	*	2	2	3	2	2	2	3
6	3	2	1	3	2	*	2	2	3	2	2	2
7	2	3	3	2	2	2	*	2	2	2	2	2
8	2	2	2	2	3	2	2	*	2	3	2	2
9	2	2	2	2	2	3	2	2	*	3	2	2
10	2	1	2	3	2	2	2	3	3	*	2	2
11	2	3	2	2	2	2	2	2	2	2	*	3
12	3	2	2	1	3	2	2	2	2	2	3	*

Approximate Personal Computer Time (8 hours).

Although the number of final schedules that need to be examined is difficult to calculate exactly, experiments have shown that the method still requires more time than is feasible. The method however, can be implemented to determine the final 4 meeting's schedules in a reasonable length of time. If one uses the greedy algorithm (Method 2) to choose the first four meetings' schedules, and then applies branch and bound for the final four months, an improvement over the previous solution can be found in about eight hours of personal computer time. The schedule should be initialized by placing team 1 in group *A* for each meeting, and by choosing the first four meetings' schedules. The above algorithm is then called starting with the fifth meeting. The results of this method produce a schedule with four 1's, forty-six 2's, and sixteen 3's, which has a cost of 20.

Method 4 (steepest descent) Another method considered starts with an arbitrary schedule and searches "neighboring" schedules to find one that is superior. If a superior schedule is found, the process is repeated by searching neighbors of the superior schedule, and so on. When no superior schedule is found, the search terminates.

This method looks at only a small percentage of the solutions, and is very dependent on the schedule used as a starting point for the search. This method also depends upon the definition of "neighboring" schedules. If the neighborhood is too large, the search will be very slow. If the neighborhood is too small, the search might miss better schedules. Although neighboring schedules can be chosen in a variety of ways, it was decided to call two schedules neighbors if they are identical for every month except one, and in that month there are exactly two teams that switch groups when going from one schedule to the other. Then each schedule has $8 \times 3 \times 4 \times 4 = 384$ neighbors. (There are 8 different months in which a neighbor may differ, and in that month, there are 3 ways to choose the two groups involved in the switch and 4 ways to choose a team from each of the groups.) To further reduce the number of neighbors to search, it was decided to examine only those neighboring schedules where the two teams that switch meet either too often or too infrequently in the

Steepest descent results

Schedule
(entries are group label assigned to team)

Team #	Meeting # 1	2	3	4	5	6	7	8
1	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
2	*B*	*B*	*B*	*B*	*B*	*A*	*B*	*A*
3	*A*	*C*	*C*	*A*	*B*	*B*	*B*	*B*
4	*B*	*A*	*B*	*A*	*C*	*B*	*A*	*C*
5	*B*	*A*	*A*	*C*	*B*	*C*	*C*	*B*
6	*C*	*B*	*C*	*C*	*C*	*A*	*A*	*B*
7	*C*	*C*	*B*	*B*	*A*	*C*	*A*	*B*
8	*C*	*B*	*A*	*C*	*A*	*B*	*B*	*C*
9	*B*	*B*	*C*	*A*	*A*	*C*	*C*	*A*
10	*C*	*C*	*A*	*B*	*C*	*B*	*C*	*A*
11	*A*	*C*	*B*	*C*	*B*	*A*	*C*	*C*
12	*A*	*A*	*C*	*B*	*C*	*C*	*B*	*C*

Cost of this schedule is 14.

Competition Matrix
(entries are # of times team plays opponent)

Team #	Opponent # 1	2	3	4	5	6	7	8	9	10	11	12
1	*	2	2	3	2	2	2	2	3	2	2	2
2	2	*	2	2	2	2	2	2	3	2	3	2
3	2	2	*	2	2	2	2	2	2	2	3	3
4	3	2	2	*	2	2	2	2	2	2	2	3
5	2	2	2	2	*	2	2	2	3	2	3	2
6	2	2	2	2	2	*	3	3	2	2	2	2
7	2	2	2	2	2	3	*	2	2	3	2	2
8	2	2	2	2	2	3	2	*	2	3	2	2
9	3	3	2	2	3	2	2	2	*	2	1	2
10	2	2	2	2	2	2	3	3	2	*	2	2
11	2	3	3	2	3	2	2	2	1	2	*	2
12	2	2	3	3	2	2	2	2	2	2	2	*

Approximate Personal Computer Time/
25 Trials (2 hours).

original schedule (similar but slower results were obtained by examining all neighbors). These neighbors would then have the greatest impact on the cost of the schedule. For each choice of a new starting schedule, the computer takes about 1 minute to examine the relevant neighbors at 4 neighbors per second.

This program was run 25 times and the results varied greatly with the choice of the initial schedule. The process terminated after approximately 30 exchanges, with an average best cost of 18. However the best overall cost was found to be 14 (one 1, thirteen 3's, and the rest 2's). Recall that the theoretical optimal cost is 12, and thus this schedule is very close to optimal.

It is doubtful that this method will locate the global minimum even if it exists, since it was shown by method 2 that solutions near the optimal one, in terms of the 7th month, are far from optimal in terms of the 8th month. Also, this method does not make use of the symmetry of the problem to eliminate possibilities. Hence of the approximately 10^{27} possibilities, only about 3000 (100 neighbors $\times$ 30 steps) neighbors were checked. This represents a very small percentage of trials.

Method 5 (annealed search) Lastly the authors considered an *annealed search*. Annealed search is a method derived from the modeling of cooling of a liquid into a crystal. If the liquid cools too fast, the crystal has imperfections. Hence the cooling must be done slowly to ensure a perfect crystal. The cooling is analogous to the descent of the search. The faster the descent the faster the crystallization takes place. Hence this method models the search as a slow descent. Annealed search is related to steepest descent in that old schedules are replaced with neighboring schedules in the hope of finding the best one. The difference lies in the choice of which neighbor is chosen as a new starting point. In annealed search, an arbitrary neighbor is chosen. If the cost of the neighboring schedule is less than the original schedule, then the old schedule is replaced with the new schedule and the process is repeated. If the neighbor's cost is greater than the cost of the original schedule, then this neighbor might still be chosen as a new starting point. This allows the process to descend more slowly and hopefully produce better results. A higher cost neighbor will be chosen if the absolute difference between the cost of the two schedules divided by an annealing

Annealed optimization results

Schedule
(entries are group label assigned to team)

Team #	Meeting # 1	2	3	4	5	6	7	8
1	*A*	*A*	*A*	*A*	*A*	*A*	*A*	*A*
2	*A*	*A*	*B*	*B*	*B*	*B*	*B*	*B*
3	*B*	*B*	*C*	*A*	*A*	*B*	*B*	*B*
4	*A*	*C*	*B*	*A*	*C*	*C*	*C*	*B*
5	*C*	*A*	*C*	*B*	*A*	*C*	*C*	*C*
6	*C*	*C*	*A*	*A*	*B*	*B*	*A*	*C*
7	*C*	*C*	*C*	*B*	*C*	*A*	*B*	*A*
8	*B*	*A*	*A*	*C*	*C*	*B*	*C*	*A*
9	*C*	*B*	*A*	*C*	*B*	*A*	*C*	*B*
10	*B*	*C*	*B*	*C*	*A*	*A*	*B*	*C*
11	*A*	*B*	*C*	*C*	*C*	*C*	*A*	*C*
12	*B*	*B*	*B*	*B*	*B*	*C*	*A*	*A*

Cost of this schedule is 14.

Competition Matrix
(entries are # of times team plays opponent)

Team #	Opponent # 1	2	3	4	5	6	7	8	9	10	11	12
1	*	2	2	2	2	3	2	3	2	2	2	2
2	2	*	3	3	2	2	2	2	2	2	1	3
3	2	3	*	2	2	2	2	2	2	3	2	2
4	2	3	2	*	2	2	2	2	2	2	3	2
5	2	2	2	2	*	2	3	2	2	2	3	2
6	3	2	2	2	2	*	2	2	3	2	2	2
7	2	2	2	2	3	2	*	2	2	3	2	2
8	3	2	2	2	2	2	2	*	3	2	2	2
9	2	2	2	2	2	3	2	3	*	2	2	2
10	2	2	3	2	2	2	3	2	2	*	2	2
11	2	1	2	3	3	2	2	2	2	2	*	3
12	2	3	2	2	2	2	2	2	2	2	3	*

Approximate Personal Computer Time/
25 trials (1 hour).

factor is less than a random number chosen for each trial [8]. The smaller the annealing factor the more times bad neighbors are rejected and the faster the descent. Although the annealing factor must be determined experimentally, it should be on the same order of magnitude as an average cost. Such a choice of the annealing factor will allow the search a reasonable chance of foregoing a local minimum in the hopes of finding a global minimum. This method has an advantage in speed over method 4 in that only one arbitrary neighbor is examined at each iteration.

This method was allowed 2000 iterations and it was tried 25 times from a variety of initial schedules. The same local minimum cost of 14 was found. However, this method performed much faster than steepest descent. Unfortunately, both methods were very sensitive to the choice of the initial schedule.

Conclusions The best results were obtained with the methods of steepest descent and annealed search (see Table below). Both methods produced comparable results in a reasonable amount of time. Annealed search must be considered superior since not only were the results obtained faster than steepest descent, but there was less analysis particular to the problem involved in the search. Further, the probabilistic nature of annealed search allows it to locate minimums starting from points where steepest descent would fail. Unfortunately, no method produced the theoretical "optimal" solution, and its existence remains unknown. It appears the best chance of locating the "optimal" solution would be to use an annealed search with a high annealing factor. This would simulate very slow cooling, and allow the method to escape from local minimums in the hopes of finding a global minimum.

Comparison of Methods

Method	Cost	Time
Global Search	$\geqslant 12$	10^{28} hrs
Greedy Algorithm	22	3 hrs
Partial Branch & Bound	20	8 hrs
Steepest Descent	14	2 hrs
Annealed Search	14	1 hr

Further experiments could also be performed using steepest descent and simulated annealing by choosing a variety of neighborhoods and making more intelligent choices for initial schedules. Other methods of attack could also be tried. One such method in the category of reverse modeling is that of neural networks [**6**, **12**]. This method (like annealed search) is based on a physical phenomenon as the rationale to solve the problem. Both methods have been used with some success to solve the well-known Traveling Salesman problem, and represent a new trend in modern general purpose techniques to solving discrete optimization problems.

Although all the above methods are capable (given nearly infinite amount of time) of solving the scheduling problem, this scheduling problem illustrates the care that must be taken before solving any problem (in finite time) on the computer. Analysis of the time needed to solve the problem is essential to produce an effective method of solution. Further, this problem illustrates that analysis can yield significant improvements in the efficiency of many methods. All methods above were improved by simple analysis.

This problem also illustrates other aspects of real world problems. Although no optimal schedule was found, good schedules were produced in reasonable time by both steepest descent and annealed search. This type of compromise between speed and accuracy is typical of solutions to real world problems, where all methods of attack must be evaluated not only by their results, but by the time needed to produce those results.

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It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all.

J.J. Sylvester, Collected Works, vol. 4, p.600.

Rubik's Tesseract

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The great popularity of Ernö Rubik's ingenious cubical puzzle led to the appearance of many variations on Rubik's idea: a $4 \times 4 \times 4$ cube, puzzles in the shape of tetrahedra and dodecahedra, etc. One natural variation that never appeared on toy store shelves is the four-dimensional version of Rubik's cube—what might be called a "Rubik's tesseract." In this paper we consider the mathematics of the $3 \times 3 \times 3 \times 3$ Rubik's tesseract. This has also been studied independently by H. R. Kamack and T. R. Keane (see [2]), Joe Buhler, Brad Jackson, and Dave Sibley.

Of course, the tesseract is somewhat harder to work with than the cube, since we can't build a physical model and experiment with it. The results described below were discovered with the aid of a simulation of the tesseract on a Macintosh computer. In this simulation the computer displays a representation of the tesseract on the screen, and the user uses a pointing device (a mouse) to ask the computer to twist sides of the tesseract. To understand the graphic representation of the tesseract used in this simulation, it might be helpful to consider first the easier problem of representing the ordinary three-dimensional Rubik's cube in a way that two-dimensional people could understand.

One way to make a two-dimensional representation of the Rubik's cube would be to imagine unfolding the surface of the cube in the familiar way illustrated in FIGURE 1. (In all of the figures in this paper, the colors of the sides of the cube are represented as black-and-white patterns. We will continue to refer to them, however, as colors.) Unfortunately, this representation would not be very useful to a two-dimensional person trying to solve the cube. The problem is that this representation doesn't show clearly which colors are attached to different sides of the same cubie. (The 27 small cubes that make up a Rubik's cube are usually called "cubies.") For example, the three "colors" attached to the cubie in the front bottom right corner of the cube in FIGURE 1 are stripes, gray, and dots, but it takes some thought to figure this out from the two-dimensional representation in FIGURE 1.

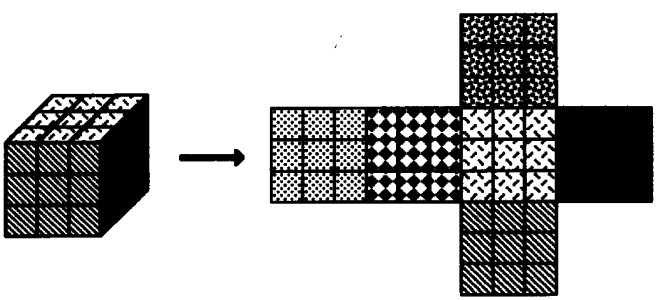

FIGURE 1

Here's a more useful way to make a two-dimensional representation of the cube. Imagine slicing the cube horizontally into three layers, and then spreading these layers out and viewing them from above (see FIGURE 2). Each layer would look like a 3×3 square, with the same four colors appearing along the sides of all three layers.

In addition, two more colors appear in the interiors of all the squares in the first and third layers; these are the colors that face down and up on the cube. (You have to imagine that you can see through the cubies in the lower layer, to see from above the dots on their bottom faces.) A two-dimensional person viewing this picture would not be able to visualize how the three square layers should be stacked up vertically to form a cube, or the directions in which the colors in the interiors of the first and third layers face on this cube. However, this representation has the advantage that the 27 small squares in this two-dimensional picture correspond to the 27 cubies of the Rubik's cube, and the colors attached to each cubie on the Rubik's cube are also attached to the corresponding square in the two-dimensional representation.

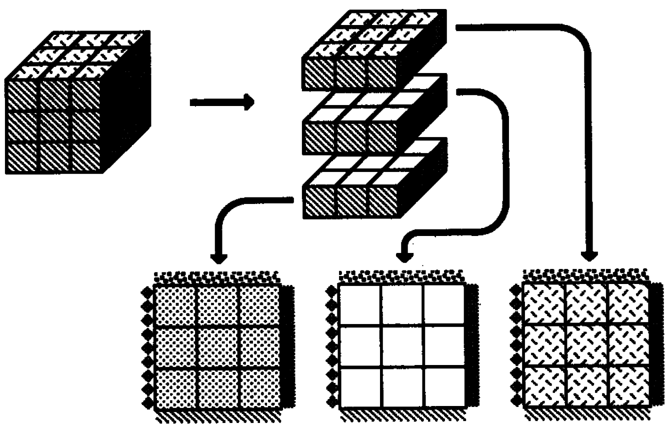

FIGURE 2

Clearly, twisting the bottom or top of the Rubik's cube corresponds in this two-dimensional representation to twisting the first or third square layer. Twists of the other four sides of the cube look somewhat more complicated in the two-dimensional representation, since they cause cubies to move between layers, and they cause some colors that start out facing up or down on the cube to face toward the sides and vice-versa. The reader might enjoy working out how these twists would look to a two-dimensional person using this representation of the Rubik's cube.

By analogy, we can imagine the $3 \times 3 \times 3 \times 3$ tesseract as three $3 \times 3 \times 3$ cubes that are stacked "up" in the fourth dimension. All three cubes have the same six colors assigned to their faces, and in addition there are two more colors assigned to the interiors of all of the cubies in the first and third cubes. (We will continue to refer to the 81 small cubes in this representation as "cubies," although each actually represents one of the 81 small tesseracts that make up the Rubik's tesseract.) We can picture it as three Rubik's cubes that are identical, except that the first and third are made out of colored plastic instead of the usual black plastic. This is illustrated in FIGURE 3; the colored stickers on the sides of the cubies are shown in this figure as being smaller than the stickers on the sides of a real Rubik's cube, to allow the colors of the plastic of the first and third cubes to show through around the edges. The computer simulation of the tesseract mentioned above displays a picture similar to FIGURE 3 on the screen of the computer.

Of course, the computer simulation also maintains internally a mathematical description of the Rubik's tesseract in which the positions of the 81 cubies are represented by assigning them coordinates in $\mathbf{R}^4$. Although this representation will not be used in the discussion below, the reader may be interested in a brief description of it. The coordinate system is set up in such a way that the centers of the 81 cubies are the points of $\mathbf{R}^4$ all of whose coordinates are -1, 0, or 1. Each cubie has an exposed side—and hence a color—facing in the direction of each dimension for which the corresponding coordinate of its center is nonzero. The twists of the "faces" of the Rubik's tesseract, which are described geometrically below, are computed mathematically by the computer by applying appropriate rotations in $\mathbf{R}^4$ to certain subsets of the tesseract.

The three-dimensional Rubik's cube has six square faces, each with a different color assigned to it; the tesseract has eight cubical "faces." The first and third cubes represent two of the faces of the tesseract, and the colors in their interiors face in opposite directions in the fourth dimension when the cubes are stacked "up" to form a tesseract. The gray stickers on the tops of all three cubes identify the top layers of the three cubes as making up another face of the tesseract. Similarly, the fronts, backs, bottoms, and left and right sides of the three cubes make up the other five faces.

The Rubik's cube contains three kinds of cubies: corner cubies, which have three colors attached to them; edge cubies, with two colors; and cubies in the centers of the faces, which have only one color. (We are ignoring the cubie in the center of the cube, which has no color attached to it and plays no role in the puzzle. In fact, readers who have taken their Rubik's cubes apart know that there actually is no cubie in the center.) Note that in some cases cubies of the same type look quite different in the two-dimensional representation of the cube described above. For example, the corners of the middle square layer and the edges of the first and third layers all represent edge cubies.

The pieces of the tesseract fall into four categories, which can be identified by the number of colors attached to them. For example, the corners of the middle cube in FIGURE 3 and the edges of the first and third cubes all have three colors attached to them, and therefore all belong to the category of three color cubies; we will call these *3C cubies*. Note that again we are ignoring the cubie in the center of the center cube in FIGURE 3, but the centers of the first and third cubes have a color assigned to their interiors, so they are 1C cubies. The reader can check by studying FIGURE 3 that there are a total of 8 1C cubies, 24 2C cubies, 32 3C cubies, and 16 4C cubies.

The tesseract can be scrambled by twisting any of its eight faces. The twists that are easiest to understand in FIGURE 3 involve rotating the first and third cubes, which we have already seen represent two of the eight faces. Since the faces are cubical rather than square, they can be rotated in many different directions. For example, we

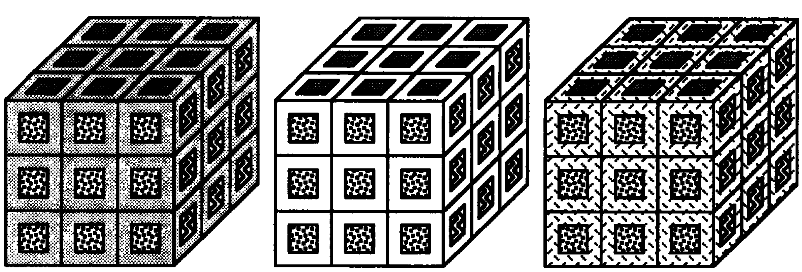

FIGURE 3

could rotate the first cube 90° around a vertical axis to bring the zigzag color on its right side to the front. Note that this rotation can be thought of as simultaneously twisting the lower, middle, and upper layers of the first cube in the same direction. Similarly, we can imagine slicing this cube into either front, middle, and back layers or left, middle, and right layers, and there are other rotations of the cube that will result in these layers being twisted simultaneously 90° in the same direction. Other rotations are also possible, such as rotations around diagonal axes, but note that they can all be accomplished by composing the three types of 90° rotations.

Now let's consider the face consisting of the right sides of all three cubes. These sides form another cubical face when the cubes are stacked "up" in the fourth dimension, but our inability to visualize this stacking makes some of the possible rotations of this face difficult to understand. However, there is one rotation of this face that is easy to understand. The remarks in the last paragraph should make it clear that there is a way of rotating this face that has the effect of simultaneously twisting the right sides of all three cubes in the same direction. Similarly, we can twist any side of the three cubes in any direction, as long as we perform the same twist simultaneously on all three cubes. There are other rotations of faces of the tesseract that are harder to understand because they move some cubies from one cube to another, and they cause the colors assigned to the interiors of some cubies to move to their surfaces, and vice-versa. Readers who have worked out how twists of the sides of the Rubik's cube would look to a two-dimensional person should be able to figure out the effects of these more complicated rotations. Fortunately, it is possible to analyze the mathematics of the tesseract without understanding in detail what these rotations do.

The key to unscrambling both the cube and the tesseract is to find sequences of twists whose net effect is to perform some simple, useful operation on the cube or tesseract. Sequences of twists are called *processes*, and the sets of all processes on the cube and tesseract form groups under composition, when processes that have the same effect are identified. It should be clear from the last paragraph that any Rubik's cube process can be applied simultaneously to all three cubes of the tesseract. However, there is a simple trick that makes it possible to perform some cube processes on *only one* cube of the tesseract, without affecting the other two cubes at all. Readers who want to solve the tesseract without getting any hints might want to try to find this trick for themselves before reading the next paragraph.

Consider the following three-step tesseract process. First rotate the first cube so that the gray stickers that start out on top end up facing to the right, then twist the right sides of all three cubes 90° clockwise, and finally undo the first rotation to return the gray side of the first cube to the top. The net effect of this process is to twist the top of the first cube and the right sides of the other two cubes 90° clockwise. Similar processes can be used to twist simultaneously the right sides of the last two cubes and any side of the first cube, and by composing such processes we can perform any cube process on the first cube, simultaneously twisting the right sides of the other two cubes some number of times. Let us define the *total twist* of a cube process to be the total number of 90° clockwise twists in the process. (A 90° counterclockwise twist can be treated as three 90° clockwise twists.) Then it should be clear that if this procedure is used on a cube process whose total twist is a multiple of 4, then the result will be a tesseract process whose net effect is to perform this cube process on the first cube of the tesseract, leaving the other two cubes unchanged. A similar procedure can be used to apply any cube process whose total twist is a multiple of 4 to any of the three cubes of the tesseract, without changing the other two cubes.

In fact, *any* cube process can be applied to the middle cube of the tesseract. The reader is invited to try proving this by finding a four-move process that twists a side of the middle cube 90°, leaving the rest of the tesseract fixed. By a *move* here we mean any reorientation of a face of the tesseract, including 180° rotations and rotations around diagonal axes. (Here's one way to approach this problem: First find a four-move process that twists the middle layer of the first cube 90°, leaving the rest of the tesseract fixed. Then figure out why this is essentially the same as the original problem.) Cube processes whose total twist is even can also be applied to the first and third cubes of the tesseract, but I know of no easy proof of this. (A somewhat indirect proof can be constructed using the analysis of possible positions of the tesseract, which is presented below. According to this analysis, the result of applying a cube process with even total twist to the first or third cube is a possible position.) The shortest tesseract process I have found that twists one face of the first cube 180° and fixes the rest of the tesseract is 35 moves long.

Fortunately, many useful cube processes have a total twist that is a multiple of 4. Some of them are commutators—elements of the process group of the form $aba^{-1}b^{-1}$—and clearly the total twist of a commutator is always a multiple of 4. For both the cube and the tesseract, it is useful to consider two kinds of processes: those that change the locations of cubies, and those that leave the locations of all cubies fixed but change the orientations of some cubies. In the first category, there are well-known commutator cube processes that change the locations of just three corner cubies or three edge cubies, leaving the locations and orientations of all other cubies fixed (see [1] and [3]); we will use the language of permutation groups and call these *corner 3-cycles* and *edge 3-cycles*. In the second category, there are commutator processes that change the orientations of two edge cubies and others that twist two corner cubies 120° in opposite directions, leaving the rest of the cube unchanged. All of these processes have a total twist that is a multiple of 4, so the technique described above can be applied to them, resulting in tesseract processes that perform 3-cycles of corners or edges of any of the three cubes, processes that flip two edge cubies of any cube, and processes that twist two corner cubies of any cube in opposite directions. It is not hard to modify these processes slightly to find processes that perform arbitrary 3-cycles of 2C, 3C, or 4C cubies, and processes that change the orientations of pairs of 2C, 3C, or 4C cubies from different cubes. At this point it may seem that the tesseract is not very different from the cube, but there are a few surprises still to come.

If we ignore the orientations of the cubies, it is not hard now to analyze which permutations of cubies can be achieved by rotating the sides of the tesseract. Consider again the rotation of the first cube 90° around a vertical axis, bringing the zigzag color on its right side to the front. Looking at the effect of this rotation on the different categories of cubies, we see that it results in a 4-cycle of 2C cubies, three disjoint 4-cycles of 3C cubies, and two disjoint 4-cycles of 4C cubies. Clearly the same would be true of any 90° rotation of any face, so each such rotation results in an odd permutation of 2C and 3C cubies and an even permutation of 4C cubies. Note that 1C cubies are not affected by any rotations, so we can ignore them from now on. Since every process can be written as a composition of 90° rotations of faces, every process must cause an even permutation of 4C cubies, and permutations of the 2C and 3C cubies that are either both even or both odd.

Since every even permutation can be written as a composition of 3-cycles, we can use the tesseract 3-cycle processes derived above to achieve any even permutations of the 2C, 3C, and 4C cubies. To reach a configuration in which the permutations of the 2C and 3C cubies are both odd, first do a 90° rotation of any face. Now the permutations of the 2C and 3C cubies required to reach the desired configuration are even, and can therefore be achieved as before. Thus the possible permutations of cubies are precisely those that consist of an even permutation of 4C cubies and permutations of 2C and 3C cubies that have the same parity.

To analyze how the orientations of cubies can be changed by rotating sides of the tesseract, it will be useful to introduce some notation. First, let us fix a numbering of the dimensions of the tesseract. We will let dimension number 1 be the front-to-back dimension, dimension 2 the left-to-right dimension, dimension 3 the top-to-bottom dimension, and of course dimension 4 will be the fourth dimension, represented in FIGURE 3 by the assignment of cubies to different cubes. We define the *dimension number* of any color to be the number of the dimension in which that color faces when the tesseract is unscrambled. For example, the gray stickers on all three cubes face up when the tesseract is unscrambled, so the dimension number of gray is 3. Remember that the colors in the interiors of the first and third cubes face in the fourth dimension, so their dimension number is 4.

Finally, in any position of the tesseract we assign to each cubie an *orientation vector* $\mathbf{c} = (c_1, c_2, c_3, c_4)$, where $c_i =$ the dimension number of the color that is facing in dimension i on this cubie. If there is no color facing in dimension i, we let $c_i = 0$. Of course, when the tesseract is unscrambled we always have either $c_i = i$ or $c_i = 0$, but when the tesseract is scrambled colors sometimes face in directions other than their original directions, so we may have $0 \neq c_i \neq i$. Note that the dimension numbers of colors are fixed, but the orientation vectors of cubies can change when sides of the tesseract are twisted.

For example, consider the top right edge cubie of the first cube in FIGURE 3. It has no color facing forwards or backwards, so the first coordinate of its orientation vector is 0, but it does have colors facing in the other three dimensions. Thus its orientation vector is $(0,2,3,4)$. Now suppose we rotate the first cube 90° around a vertical axis, bringing this cubie to the front. Then the zigzag color on its right side will move to the front, and in this new position there will be no color on this cubie facing toward the left or right. Since the dimension number of the zigzag color is 2, the orientation vector of this cubie after the twist will be $(2,0,3,4)$. Note that the first two coordinates of the orientation vector have been exchanged by this twist. In fact, the reader should be able to verify that this twist causes the first two coordinates of the orientation vectors of all the cubies it affects to be exchanged. The same is true of other 90° rotations of faces, except that different rotations may cause different pairs of coordinates to be exchanged.

Using this notation, we can now analyze the possible orientations of cubies when the tesseract is scrambled. Let us first consider the orientations of the 2C cubies. Each of these cubies has only two nonzero entries in its orientation vector. We will call a 2C cubie *sane* if the nonzero entries in its orientation vector appear in increasing order, and *flipped* otherwise. Of course, all 2C cubies are sane before the tesseract is scrambled.

When the first cube is rotated 90° around a vertical axis, the four 2C cubies in its middle layer are affected. All of them have an interior color that faces in dimension 4 (and thus the fourth coordinates of their orientation vectors are nonzero), and in addition two have colors facing in dimension 1 and two have colors facing in dimension 2. The twist exchanges the first two coordinates of the orientation vectors of all four cubies. In general, for any 90° rotation of a face of the tesseract there will be only four 2C cubies affected, and there will be distinct numbers i, j, and k such that the twist exchanges coordinates i and j of the orientation vectors of the cubies, all four of the cubies have nonzero kth coordinates in their orientation vectors, and in addition two have nonzero ith coordinates and two have nonzero jth coordinates. Clearly if k is between i and j, then all four cubies will have their sanities switched by this twist, and otherwise their sanities will be unchanged. It follows that the number of flipped 2C cubies will be even in all possible positions of the tesseract. Thus if we are told the locations and orientations of all 2C cubies except one in some scrambled position of the tesseract, we can deduce the orientation of the last 2C cubie. Since we have already seen that there are processes that flip the 2C cubies two at a time, this is the only restriction on the possible orientations of the 2C cubies.

Consider now any 3C cubie. Before the tesseract is scrambled it has an orientation vector $\mathbf{c} = (c_1, c_2, c_3, c_4)$ that has three nonzero entries. After some process has been executed, it will have a new orientation vector $\mathbf{c}'$ that is a permutation of $\mathbf{c}$. We will call the orientation of this cubie *even* or *odd*, according to whether this permutation is even or odd. Clearly the orientations of all 3C cubies are even before the tesseract is scrambled, and a 90° twist of a face affects 12 3C cubies, transposing two coordinates of their orientation vectors and therefore changing the parities of their orientations. Thus in any position of the tesseract there must be an even number of 3C cubies whose orientations are odd. Note that, unlike the restriction given above on the orientations of 2C cubies, this does not give us enough information to determine the orientation of a 3C cubie, even if we know the locations and orientations of all other cubies. For example, consider a process that leaves the locations and orientations of all cubies fixed, except perhaps for one 3C cubie. We know the orientation of this cubie must be even, but the three colors attached to this cubie can be permuted in $3! = 6$ ways, and 3 of these are even permutations. Thus, either there are other restrictions on the possible orientations of the 3C cubies, or there must be processes which change the orientation of a single 3C cubie, leaving the rest of the tesseract fixed. Experience with the Rubik's cube suggests that the first of these possibilities is the most likely, but in fact the second is correct.

To see how to construct a process that changes the orientation of a single 3C cubie, recall that we already know how to flip two edge cubies on the first or third cube. By modifying these processes we can in fact exchange any two colors on any two 3C cubies, fixing the rest of the tesseract. Now let C_1, C_2, and C_3 be any three 3C cubies. Let p be a process that exchanges the first two colors on C_1 and any two colors on C_2, and let q be a process that exchanges the last two colors on C_1 and any two colors on C_3. We let the reader verify that the commutator $pqp^{-1}q^{-1} = (pq)^2$ performs a 3-cycle on the colors attached to C_1, and has no net effect on C_2 or C_3. (The reader might enjoy trying to find a more efficient process that produces the same result. My shortest solution takes 20 moves.) Thus any even permutation can be performed on the colors attached to any 3C cubie. Combining this with the fact that any two colors on any two 3C cubies can be exchanged, we can conclude that the only restriction on the orientations of 3C cubies is the one we have already stated, that the number of 3C cubies with odd orientations must be even.

Finally, consider the 4C cubies. The four colors attached to any 4C cubie can be permuted in $4! = 24$ ways, but half of these can be ruled out immediately by a parity argument. To prove this, we set up a four-dimensional coordinate system in which the coordinates of the centers of the 4C cubies are $(\pm 1, \pm 1, \pm 1, \pm 1)$. We define the *sign* of the location of a 4C cubie to be the product of the coordinates of its center. Thus, the location is positive if there are an even number of -1's in the coordinates, and negative if there are an odd number of -1's. As before, we also call the orientation of a 4C cubie even or odd depending on how its orientation vector has been permuted. The reader can now check that a 90° rotation of a face affects eight 4C cubies, changing both the sign of the location and the parity of the orientation of each. Therefore if a cubie's location has the same sign as it had before the tesseract was scrambled, then its orientation must be even, and if not, its orientation must be odd. In particular, any process that does not change the locations of any 4C cubies can only perform even permutations on their orientation vectors.

This still leaves us with 12 possible orientations for each 4C cubie. Using processes we have already discussed we can simultaneously put 15 of the 16 4C cubies in any of these 12 orientations, as follows. We have already found processes that twist two corners of the first or third cube 120° in opposite directions. Variations on these processes will allow us to perform any 3-cycle on the orientation vector of any 4C cubie, simultaneously performing a 3-cycle on the orientation vector of some other 4C cubie as well. Combining these processes we can therefore perform any combination of even permutations on the orientation vectors of all the 4C cubies except one. We must still determine the possible orientations for this last 4C cubie.

It will turn out that there are fewer than 12 possible orientations for the last 4C cubie, but based on our discussion of 3C cubies the reader can probably guess that there will be more than one possibility. In fact, the proof of this is very similar to the proof for 3C cubies. Let C_1, C_2, and C_3 be three 4C cubies. Let p be a process that performs a 3-cycle on the first three coordinates of the orientation vector of C_1, simultaneously permuting the orientation vector of C_2, and let q be a process that performs a similar 3-cycle on the last three coordinates of the orientation vector of C_1, simultaneously permuting the orientation vector of C_3. Then the commutator $pqp^{-1}q^{-1}$ transposes the first and last coordinates, and also the second and third coordinates, of the orientation vector of C_1, leaving the rest of the tesseract fixed. Similarly we can transpose any two disjoint pairs of coordinates of the orientation vector of any 4C cubie. (Again, there are more efficient ways of accomplishing this. My shortest process takes 16 moves.)

Let us say that two orientation vectors are *similar* if either they are equal, or we can transpose two disjoint pairs of coordinates of one to get the other. It is easy to check that this is an equivalence relation, and each equivalence class has four elements. To complete the analysis of the possible orientations of 4C cubies, it will be useful to choose representatives of these equivalence classes, which we do as follows. We say that an orientation vector $\mathbf{c} = (c_1, c_2, c_3, c_4)$ is *normal* if $c_4 = 4$, and we define the *normal form* of $\mathbf{c}$ to be the unique vector $\bar{\mathbf{c}} = (\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$ such that $\bar{\mathbf{c}}$ is similar to $\mathbf{c}$ and $\bar{\mathbf{c}}$ is normal. Since we have shown that the orientation of any 4C cubie can be changed to any other similar orientation without changing the rest of the tesseract, it suffices now to determine the possible normal forms for the orientation vector of the last 4C cubie.

For future use we observe that if $\mathbf{c}$ and $\mathbf{c}'$ are similar orientation vectors and we transpose the same pair of coordinates of both, then the resulting vectors are still similar. Therefore, if we transpose the first and second, first and third, or second and third coordinates of $\mathbf{c}$, the normal form of the resulting vector can be found by performing the same transposition of $\bar{\mathbf{c}}$. Transposing the third and fourth coordinates of $\mathbf{c}$ yields the same result as transposing the first and second coordinates of the similar vector (c_2, c_1, c_4, c_3), and therefore the normal form of the resulting vector can be found by transposing the first and second coordinates of $\bar{\mathbf{c}}$. Similarly it can be shown that any transposition of two coordinates of $\mathbf{c}$ causes some two of the first three coordinates of the normal form of $\mathbf{c}$ to be transposed.

For an orientation vector $\mathbf{c}$ that is normal we define the *twist* of $\mathbf{c}$ to be the value of i for which $c_i = 3$. If $\mathbf{c}$ is not normal, we define the twist of $\mathbf{c}$ to be the twist of its normal form—i.e., the unique i for which $\bar{c}_i = 3$. The twist of a cubie is the twist of its orientation vector. We are now ready to state the last restriction on the orientations of 4C cubies. We claim that the sum of the twists of the cubies in positive locations and the sum of the twists of cubies in negative locations are always congruent mod 3. Equivalently, if we define the *signed twist* of a cubie to be equal to the twist of the cubie times the sign of its location, then the sum of the signed twists of all cubies is always a multiple of 3.

Clearly, before the tesseract is scrambled, all 4C cubies have orientation vector $(1, 2, 3, 4)$, which is normal, so all 4C cubies have a twist of 3. Thus the sum of the signed twists of all cubies is 0. Now consider any 90° rotation of a face of the tesseract. Recall that such a rotation transposes the same two coordinates of the orientation vectors of eight 4C cubies, simultaneously reversing the signs of their locations. As we have already observed, the effect of this operation on the normal forms of the orientation vectors of these cubies will be to cause some two of the first three coordinates of these normal forms to be transposed. Now consider the effect of this transposition on the twists of these cubies. The reader can easily check that if the twist of some normal orientation vector is t, then transposing the first two coordinates of this vector gives a normal orientation vector whose twist is congruent to $-t \bmod 3$. Similarly, transposing the first and third coordinates results in a twist congruent to $1 - t \bmod 3$, and transposing the second and third gives a twist congruent to $2 - t \bmod 3$. Thus rotating any face 90° will perform one of these three transformations on the twists of the eight cubies affected by the rotation.

Finally, we consider the effect of a 90° rotation on the signed twists of the 4C cubies affected. According to the last paragraph, for each such rotation there is a constant k such that each affected cubie with a twist of t before the rotation has a twist congruent to $k - t \bmod 3$ after rotation. Since the sign of the location of each of these cubies is reversed by the rotation, a cubie in a positive location whose twist is t has its signed twist changed from t to $t - k \bmod 3$, while if the location is negative then the signed twist changes from $-t$ to $k - t \bmod 3$. Since half of the signed twists are decreased by k and half are increased by $k \bmod 3$, the sum of the signed twists is unchanged mod 3. This proves our claim that the sum of the signed twists of the 4C cubies is a multiple of 3 in all possible positions of the tesseract.

If we are told the locations and orientations of 15 of the 16 4C cubies, we can now determine what orientations are possible for the last 4C cubie. From the restrictions derived above we can determine both the parity and the twist of the orientation. If the orientation vector $\mathbf{c}$ of this cubie is normal then we know $c_4 = 4$, and the twist of the cubie tells us the value of i for which $c_i = 3$. The values of the other two entries of the orientation vector are then determined by the parity of the orientation. Thus there is only one possible normal orientation vector for the last 4C cubie. Since any orientation vector is possible if and only if its normal form is, the possible orientations are just the four which are similar to this normal orientation.

We have now given a complete analysis of the possible scrambled positions of the tesseract. Using this analysis we can compute the number of such positions—i.e., the order of the process group. Considering first the locations of the cubies and ignoring their orientations, the 24 2C cubies, 32 3C cubies, and 16 4C cubies can be permuted in $24! \times 32! \times 16!$ ways. But only even permutations of the 4C cubies are possible, and the parities of the permutations of the 2C and 3C cubies must be the same. Thus the number of these permutations that can be achieved by rotating the sides of the tesseract is $(24! \times 32!)/2 \times 16!/2$. For each of these permutations, 23 of the 2C cubies can have either of two orientations, with the orientation of the last 2C cubie then being determined. Thirty-one of the 3C cubies can have any of six orientations, with the orientation of the last 3C cubie being restricted by its parity to only three possibilities, and 15 of the 4C cubies have 12 possible orientations, with the last having only four possibilities. Thus the total number of positions of the tesseract is:

$$
\begin{aligned}
&(24!\times 32!)/2\times 16!/2\times 2^{23}\times 6^{31}\times 3\times 12^{15}\times 4\\
&=1{,}756{,}772{,}880{,}709{,}135{,}843{,}168{,}526{,}079{,}081{,}025{,}059{,}614,\\
&\quad 484{,}630{,}149{,}557{,}651{,}477{,}156{,}021{,}733{,}236{,}798{,}970{,}168,\\
&\quad 550{,}600{,}274{,}887{,}650{,}082{,}354{,}207{,}129{,}600{,}000{,}000{,}000,\\
&\quad 000\\
&\cong 1.76\times 10^{120}.
\end{aligned}
$$

For comparison, we note that the number of positions for the Rubik's cube is a measly 4.33×10^{19}.

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The Catalan Numbers and Pi

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The Catalan numbers $\binom{2n-2}{n-1}/n$, $n=1,2,\ldots,$ arise naturally in many problems of *discrete* mathematics. For a vigorous discussion of some of these problems see the recent article of R. B. Eggleton and R. K. Guy [3]. The universal constant π is truly ubiquitous throughout mathematics and the empirical sciences. However, the constant occurs most frequently in elementary calculus, which is the cornerstone of *continuous* mathematics. The following series representation of $1/\pi$ relates the Catalan numbers and π in a curious manner.

$$\frac{1}{\pi}=\frac{3}{16}+\frac{9}{4}\sum_{k=1}^{\infty}\left\{\frac{\binom{2k-2}{k-1}}{k}\right\}^{2}\frac{4k^{2}-1}{2^{4k}(k+1)^{2}}. \tag{1}$$

In [4, pp. 36–38] Ramanujan presented 17 series representations of $1/\pi$, and within the confines of elliptic function theory proved three of these. Apparently, Ramanujan had very little interest in combinatorics, and accordingly made no attempt to interpret the terms of the series in terms of interesting combinatorial objects. However, several of his series have terms that involve the central binomial coefficients $\binom{2n}{n}$, $n=0,1\ldots$. Unlike the series representations of Ramanujan our representation (1) requires no advanced machinery for its justification. In fact, all of the tools can be found in any good elementary calculus textbook.

having only four possibilities. Thus the total number of positions of the tesseract is:

$$\begin{aligned}&(24!\times 32!)/2\times 16!/2\times 2^{23}\times 6^{31}\times 3\times 12^{15}\times 4\\ &=1{,}756{,}772{,}880{,}709{,}135{,}843{,}168{,}526{,}079{,}081{,}025{,}059{,}614,\\ &\quad 484{,}630{,}149{,}557{,}651{,}477{,}156{,}021{,}733{,}236{,}798{,}970{,}168,\\ &\quad 550{,}600{,}274{,}887{,}650{,}082{,}354{,}207{,}129{,}600{,}000{,}000{,}000,\\ &\quad 000\\ &\cong 1.76\times 10^{120}.\end{aligned}$$

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To prove (1) we observe, first of all, that for $0 \leqslant x \leqslant 1$,

$$\int_0^x t\sqrt{1-t^2}\,dt = \frac{1}{3}\left\{1-(1-x^2)^{3/2}\right\}. \tag{2}$$

On the other hand, we expand the integrand and integrate termwise to get

$$\int_0^x t\sqrt{1-t^2}\,dt = \frac{x^2}{2} - \sum_{k=1}^{\infty} \frac{\binom{2k-2}{k-1}}{k2^{2k-1}} \frac{x^{2k+2}}{2k+2}$$

$$= \frac{x^2}{2} - \sum_{j=2}^{\infty} \frac{\binom{2j-4}{j-2}}{(j-1)2^{2j-3}} \frac{x^{2j}}{2j}. \tag{3}$$

Now, (2) and (3) imply

$$\frac{1}{3}\left\{1-(1-x^2)^{3/2}\right\} = \frac{x^2}{2} - \sum_{j=2}^{\infty} \frac{\binom{2j-4}{j-2}}{(j-1)2^{2j-3}} \frac{x^{2j}}{2j}. \tag{4}$$

(Note that in (2) and therefore (3), $x = 1$ represents the limiting case.) In (4) we let $x \to \sin x$ and integrate the resulting equation from 0 to $\pi/2$ to get

$$\frac{\pi}{6} - \frac{2}{9} = \int_0^{\pi/2} \frac{(\sin x)^2}{2}\,dx - \sum_{j=2}^{\infty} \frac{\binom{2j-4}{j-2}}{(j-1)2^{2j-3}} \int_0^{\pi/2} \frac{\sin^{2j} x}{2j}\,dx.$$

At this juncture we appeal to Wallis' formula [2, p. 223] in the form

$$\int_0^{\pi/2} \sin^{2j} x\,dx = \frac{\pi}{2} \cdot \frac{\binom{2j}{j}}{2^{2j}}, \qquad j = 1, 2, \ldots,$$

to write

$$\frac{\pi}{6} - \frac{2}{9} = \frac{\pi}{8}\left\{1 - \sum_{k=1}^{\infty} \frac{\binom{2k-2}{k-1}^2}{2^{4k-2}} \cdot \frac{(2k-1)(2k+1)}{k^2(k+1)^2}\right\}.$$

Finally, we divide the foregoing equation by $\pi/8$ and simplify to obtain (1).

Concluding Remarks. J. M. Borwein and P. B. Borwein [1, pp. 177–187] have recently shown that all of Ramanujan's series representations of $1/\pi$ can be justified within the theory of hypergeometric functions. These authors have utilized several of these series representations to decimally approximate π to any desired number of digits.

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Sums of Powers of Integers

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The sum $1^k + 2^k + \cdots + n^k$ of the first n positive integers each raised to a nonnegative integer power k arises often. Reflecting the variety of uses of such sums, many techniques for obtaining an expression for the sum and several forms for the resulting polynomial in n, e.g., expanded, in factored form, or in terms of Bernoulli polynomials, are well known.

Denoting the sum $1^k + 2^k + \cdots + n^k$ by $S(n; k)$, we have

$S(n;0) = n$	1				
$S(n;1) = (1/2)n^2 + (1/2)n$	1/2	1/2			
$S(n;2) = (1/3)n^3 + (1/2)n^2 + (1/6)n$	1/3	1/2	1/6		
$S(n;3) = (1/4)n^4 + (1/2)n^3 + (1/4)n^2$	1/4	1/2	1/4	0	
$S(n;4) = (1/5)n^5 + (1/2)n^4 + (1/3)n^3 - (1/30)n$	1/5	1/2	1/3	0	−1/30.

Focusing only on the coefficients of these polynomials, as shown on the right above, some patterns seem clear and it is reasonable to guess that one row of the triangular array of numbers might be determined by the previous row. This article establishes that such a guess is in fact correct and then exploits that relationship to develop an algorithm for obtaining $S(n; k+1)$ from $S(n; k)$.

An analogy may be useful. One way to expand $(x+y)^k$ as a polynomial in x and y is to read the coefficients from the $(k+1)$st row of Pascal's triangle, the construction of which proceeds simply and mechanically from one row to the next. Moreover, the method can be justified in general using only elementary concepts. Similarly, this article presents an algorithm for generating the coefficients of $S(n; k)$, expressed as a polynomial in n of degree $k+1$. The coefficients appear in a triangular array, the construction of which proceeds from one row to the next. Moreover, the validity of the algorithm can be established using only ideas typically available to lower-division mathematics students.

For completeness, most details, even those readily available elsewhere in the literature, are included. First, the fact that $S(n; k)$ is actually a polynomial in n of degree $k+1$ is proven. Then, a recursion relationship among these polynomials is established, from which, finally, the algorithm follows. The proofs use the binomial theorem, a simple application of mathematical induction, differentiation of polynomials, and the fact that two polynomials agreeing at infinitely many points must be identical.

Unless otherwise stated, throughout this note n is a positive integer, k a nonnegative integer, and x an arbitrary real number. We set $S(0; k) = 0$ for k positive.

From the binomial theorem it immediately follows that

$$(x+1)^n - x^n = \sum_{j=1}^{n} \binom{n}{j} x^{n-j}.$$

We use this observation in the following result, many different proofs of which exist; see, e.g., [**1** and **2**].

PROPOSITION 1.

$$(n+1)^{k+1}-1=\sum_{j=1}^{k+1}\binom{k+1}{j}S(n;k+1-j).$$

Proof.

$$(n+1)^{k+1}-1=\sum_{i=0}^{n-1}\left[(n-i+1)^{k+1}-(n-i)^{k+1}\right] \quad \text{(telescoping sum)}$$

$$=\sum_{i=0}^{n-1}\left[\sum_{j=1}^{k+1}\binom{k+1}{j}(n-i)^{k+1-j}\right]$$

$$=\sum_{j=1}^{k+1}\left[\binom{k+1}{j}\sum_{i=0}^{n-1}(n-i)^{k+1-j}\right]$$

$$=\sum_{j=1}^{k+1}\binom{k+1}{j}S(n;k+1-j).$$

PROPOSITION 2. *$S(n;k)$ is a polynomial in n of degree $k+1$ with constant term equal to zero.*

Proof. We prove this by induction on k. Since $S(n;0)=n$, the assertion is valid for $k=0$. Assume it to be valid for $k=0,1,\dots,m-1$. Then, by Proposition 1 and the induction hypothesis,

$$S(n;m)=\frac{1}{m+1}\left\{(n+1)^{m+1}-1-\left[\sum_{j=2}^{m+1}\binom{m+1}{j}S(n;m+1-j)\right]\right\},$$

from which it is clear that $S(n;m)$ is a polynomial in n of degree $m+1$. Since $S(0;k)=0$, the constant term must equal zero, so the assertion is valid for $k=m$. By induction, the result follows.

Define the coefficients of $S(n;k)$, expressed as a polynomial in n of degree $k+1$, by $S(n;k)=a_{k,k+1}n^{k+1}+a_{k,k}n^k+\cdots+a_{k,1}n$. Let $P_k(x)$ be the polynomial of degree $k+1$ obtained by replacing n by x in $S(n;k)$, i.e., $P_k(x)=a_{k,k+1}x^{k+1}+a_{k,k}x^k+\cdots+a_{k,1}x$. Finally, let $\Delta f(x)=f(x+1)-f(x)$ and $Df(x)=f'(x)$.

PROPOSITION 3. (a) $P_k(x+1)-P_k(x)=(x+1)^k$.
(b) *$D\Delta f(x)=\Delta Df(x)$ for any differentiable function f.*
(c) *If P is a polynomial and $\Delta P(x)=0$ for all x, then P is a constant.*
(d) $a_{k,k+1}+a_{k,k}+\cdots+a_{k,1}=1$.

Proof. By the definitions of S and P_k, the equality in (a) is valid for x any positive integer; (a) then follows since two polynomials can be identical at infinitely many points only if they are equal. Each side of the equality in (b) equals $f'(x+1)-f'(x)$. (c) is valid since $P(m)=P(0)$ for all integers m, which only can be valid for a polynomial if it is constant. Finally, (d) follows since $S(1;k)=1$.

THEOREM. $kP_{k-1}(n) = P_k'(n) - P_k'(0)$.

Proof.

$$\begin{aligned}
\Delta DP_k(x) &= D\Delta P_k(x), && \text{by Proposition 3(b)},\\
&= D[P_k(x+1) - P_k(x)],\\
&= D\left[(x+1)^k\right], && \text{by Proposition 3(a)},\\
&= k(x+1)^{k-1},\\
&= k[P_{k-1}(x+1) - P_{k-1}(x)], && \text{by Proposition 3(a)},\\
&= k\Delta P_{k-1}(x)\\
&= \Delta[kP_{k-1}(x)],
\end{aligned}$$

i.e., $\Delta DP_k = \Delta kP_{k-1}$, or equivalently, $\Delta[DP_k - kP_{k-1}](x) = 0$ for all x. So by Proposition 3(c), $P_k'(x) - kP_{k-1}(x)$ is constant. Since $P_{k-1}(0) = 0$, the constant value must be $P_k'(0)$. Setting $x = n$ and rearranging terms, we obtain $kP_{k-1}(n) = P_k'(n) - P_k'(0)$.

To obtain the coefficients $a_{k,j}$, for $j = 2,\dots,k+1$, of P_k, and hence those of $S(n;k)$, from the coefficients $a_{k-1,j}$ of P_{k-1}, consider the equation given in the Theorem and simplify to obtain

$$\sum_{j=1}^{k} ka_{k-1,j}n^j = \sum_{j=1}^{k} (j+1)a_{k,j+1}n^j,$$

from which, equating like powers of n, it follows that

$$a_{k,j+1} = \frac{k}{j+1}a_{k-1,j} \quad \text{for } j = 1,2,\dots,k.$$

Moreover, $a_{k,1} = 1 - (a_{k,k+1} + a_{k,k} + \cdots + a_{k,2})$ by Proposition 3(d). These latter two equations, together with $a_{0,1} = 1$, which follows since $S(n;0) = n$, determine the $a_{k,j}$ for $k = 0,1,2,\dots$, and $j = 1,2,\dots,k+1$. Thus we obtain the following algorithm for generating the triangular array of numbers, the kth row of which contains the coefficients of $S(n;k-1)$.

Algorithm

The first row contains the number 1.

The kth row, for $k > 1$, contains k numbers computed as follows.

a) For $j = 1,2,\dots,k-1$, the jth number in row k is obtained by multiplying the jth number in row $k-1$ by $(k-1)/(k+1-j)$.
b) The kth number in row k is 1 minus the sum of all the other numbers in row k, i.e., the sum of the numbers in each row is one.

For $k \geqslant 1$, the jth number in row k, for $j = 1,\dots,k$, is the coefficient of the n^{k+1-j} term of $S(n;k-1)$ expressed as a polynomial in n of degree k.

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Supermultiplicative Inequalities for the Permanent of Nonnegative Matrices

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A fact about the determinant (abbreviated det) that is usually taught early and is very useful later is that $\det(AB)=\det(A)\det(B)$, where A and B are $n\times n$ matrices and n is a finite positive integer. This multiplicative identity for the determinant is related to a host of generalizations. In some generalizations, det is replaced by another real-valued or matrix-valued function f with matrix argument, where f is related in some way to the determinant. In some generalizations, the equality is replaced by an inequality. When the function f that replaces det satisfies $f(AB)\geqslant f(A)f(B)$, f is said to be supermultiplicative; when $f(AB)\leqslant f(A)f(B)$, f is said to be submultiplicative. Clearly, det is both supermultiplicative and submultiplicative.

The purpose of this note is to consider two well-known, apparently unrelated supermultiplicative functions of nonnegative matrices and to show that they are special cases of a natural, more general supermultiplicative function. All of these functions may be viewed as relatives of the determinant. Further, these supermultiplicative functions are surprisingly useful in the theory of products of random matrices. An application to products of random matrices is sketched at the end of this note.

For a fixed finite positive integer n, an $n\times n$ matrix with all nonnegative real elements will be called a nonnegative matrix. The well-known supermultiplicative functions of a nonnegative matrix to be considered here are the diagonal elements and the permanent (Theorems A and B).

THEOREM A. *If $A=(a_{ij})$ and $B=(b_{ij})$ are nonnegative matrices, then the ith diagonal element $(AB)_{ii}$ of AB is related to the ith diagonal elements a_{ii} of A and b_{ii} of B by $(AB)_{ii}\geqslant a_{ii}b_{ii}$, for $i=1,\dots,n$.*

Proof. $(AB)_{ii}=\sum_{j=1}^{n}a_{ij}b_{ji}\geqslant a_{ii}b_{ii}$.

Recall that $\operatorname{per}(A)$, the permanent of A, is a determinant that thinks positively, i.e., if $\sigma=(\sigma(1),\dots,\sigma(n))$ is a permutation of $(1,\dots,n)$, then $\operatorname{per}(A)=\sum_\sigma a_{1,\sigma(1)}a_{2,\sigma(2)}\cdots a_{n,\sigma(n)}$ where the summation runs over all permutations σ. Minc gives an encyclopedic account of permanents **[6]**.

THEOREM B (Brualdi 1966). *If A and B are nonnegative, then* $\operatorname{per}(AB)\geqslant\operatorname{per}(A)\operatorname{per}(B)$.

Brualdi's (1966) proof of Theorem B is elementary. In outline, every term in $\operatorname{per}(A)\operatorname{per}(B)$ appears as a term of $\operatorname{per}(AB)$, and the other terms of $\operatorname{per}(AB)$ are nonnegative.

The previously known Theorems A and B are both special cases of a more general set of inequalities involving the permanent.

For $k=1,\dots,n$, define the kth *permanent-compound* of A (any matrix over a field will do, not necessarily a nonnegative matrix) as an $\binom{n}{k}\times\binom{n}{k}$ matrix $A_{[k]}$ with elements constructed as follows.

Let $Q_{k,n}=\{(i_1,i_2,\dots,i_k)|1\leqslant i_1<i_2<\cdots<i_k\leqslant n\}$ and choose an ordering, say lexicographic, of the k-tuples in $Q_{k,n}$. By a slight abuse of notation, the elements of $A_{[k]}$ will be indexed by pairs $(\underline{i},\underline{j})\in Q_{k,n}\times Q_{k,n}$ rather than by pairs of integers. For $\underline{i}=(i_1,\dots,i_k)\in Q_{k,n}$, $\underline{j}=(j_1,\dots,j_k)\in Q_{k,n}$, define $A[\underline{i};\underline{j}]$ to be the $k\times k$ matrix that contains the elements in the intersections of rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$. Then $A_{[k]}$ is the $\binom{n}{k}\times\binom{n}{k}$ matrix with $(\underline{i},\underline{j})$ element $(A_{[k]})_{\underline{i}\underline{j}}=\operatorname{per}(A[\underline{i};\underline{j}])$.

As examples, if we assume the lexicographic ordering of the elements of $Q_{k,n}$ is chosen, $A_{[1]}=A$ and if $n=3$, then

$$A_{[2]}=\begin{pmatrix} a_{11}a_{22}+a_{21}a_{12} & a_{11}a_{23}+a_{21}a_{13} & a_{12}a_{23}+a_{22}a_{13}\\ a_{11}a_{32}+a_{31}a_{12} & a_{11}a_{33}+a_{31}a_{13} & a_{12}a_{33}+a_{32}a_{13}\\ a_{21}a_{32}+a_{31}a_{22} & a_{21}a_{33}+a_{31}a_{23} & a_{22}a_{33}+a_{32}a_{23}\end{pmatrix}$$

Regardless of the ordering of $Q_{k,n}$, $A_{[n]}=\operatorname{per}(A)$. Changing the ordering of the elements of $Q_{k,n}$ simultaneously permutes the rows and columns of the permanent-compound matrix, leaving the diagonal elements $(A_{[k]})_{\underline{i}\underline{i}}$ on the diagonal. The kth permanent-compound of the $n\times n$ identity matrix is the $\binom{n}{k}\times\binom{n}{k}$ identity matrix. For any scalar c, $(cA)_{[k]}=c^kA_{[k]}$. If A^* denotes the conjugate transpose of A, then $(A_{[k]})^*=(A^*)_{[k]}$ because $\operatorname{per}(A)=\operatorname{per}(A^T)$, where A^T is the transpose of A ([**6**], p. 16) and the conjugate of a product of two complex numbers is the product of their conjugates.

Theorem A is the special case of the following Theorem 1 when $k=1$ and Theorem B is the special case when $k=n$. Thus Theorem 1 is a natural generalization of Theorems A and B.

THEOREM 1. *If A and B are nonnegative matrices, then for $k=1,\dots,n$, and all $\underline{i}\in Q_{k,n}$, $((AB)_{[k]})_{\underline{i}\underline{i}}\geqslant(A_{[k]})_{\underline{i}\underline{i}}(B_{[k]})_{\underline{i}\underline{i}}$.*

Before proving Theorem 1, note that Theorem 1 is unaffected by the ordering chosen for $Q_{k,n}$ because the theorem deals only with diagonal elements of the permanent-compound.

To prove Theorem 1, two easy lemmas are needed. Let $(AB)[\underline{i};\underline{j}]$ denote the $k\times k$ matrix that contains the elements in the intersections of rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$ of the product matrix AB.

LEMMA 1. *For any $\underline{i}\in Q_{k,n}$ and any nonnegative A,B, $(AB)[\underline{i};\underline{i}]\geqslant A[\underline{i};\underline{i}]B[\underline{i};\underline{i}]$, where the inequality applies elementwise.*

Proof. For $1\leqslant g,h\leqslant k$, let i_g and i_h denote any two elements of $\underline{i}$. Then

$$(AB)_{i_gi_h}=\sum_{m=1}^{n}a_{i_gm}b_{mi_h}\geqslant\sum_{p=1}^{k}a_{i_gi_p}b_{i_pi_h}=\left(A[\underline{i};\underline{i}]B[\underline{i};\underline{i}]\right)_{i_gi_h}.$$

LEMMA 2. *For any nonnegative A,B, if $A\geqslant B$ (elementwise), then* $\operatorname{per}(A)\geqslant\operatorname{per}(B)$.

This is obvious.

Proof of Theorem 1. Let $\underline{i}$ denote the ith element of $Q_{k,n}$ in the chosen ordering. Then for any nonnegative A,B,

$$\begin{aligned}((AB)_{[k]})_{ii}&=\operatorname{per}\{(AB)[\underline{i};\underline{i}]\}\\ &\geqslant\operatorname{per}\{A[\underline{i};\underline{i}]B[\underline{i};\underline{i}]\}\quad\text{(by Lemmas 1 and 2)}\\ &\geqslant\operatorname{per}\{A[\underline{i};\underline{i}]\}\operatorname{per}\{B[\underline{i};\underline{i}]\}\quad\text{(by Theorem B)}\end{aligned}$$

$$= (A_{[k]})_{\underline{i}\underline{i}} (B_{[k]})_{\underline{i}\underline{i}}.$$

THEOREM 2. *If A and B are nonnegative, then for $k = 1, \ldots, n$, and $(\underline{i}, \underline{j}) \in Q_{k,n} \times Q_{k,n}$,*

$$((AB)_{[k]})_{\underline{i}\underline{j}} \geqslant \max\left\{ (A_{[k]})_{\underline{i}\underline{h}} (B_{[k]})_{\underline{h}\underline{j}} | \underline{h} \in Q_{k,n} \right\}.$$

The proof parallels the proof of Theorem 1 exactly. The supermultiplicative inequality in Theorem 1 describes the special case of Theorem 2 when $\underline{i} = \underline{h} = \underline{j}$.

The permanent-compound is closely related to similarly defined objects, some of which have similar properties, e.g., the determinant-compound or adjugate matrix ([5], pp. 86–87), the induced matrix ([6], p. 87) and the combinatorial compound matrix [2]. The determinant-compound matrix is defined in the same way as the permanent-compound matrix except that per is replaced by det. The induced matrix is defined in terms of permanents of submatrices of a given matrix, but the definition is a bit more elaborate than that of the permanent-compound matrix given above. Like the simple determinant, both the determinant-compound matrix and the induced matrix preserve the product of matrices, i.e., the determinant-compound matrix of a product of matrices is the product of the determinant-compound matrices, and similarly for the induced matrices ([5], pp. 86–87; [6], p. 87). Although I know of no earlier definition of the permanent-compound matrix, I make no claim to be the first to consider it.

To conclude, I sketch the application of supermultiplicative functions to the theory of products of random matrices [4]. To avoid complications, I will describe only a special case of available results. Even so, this sketch presumes some familiarity with probability theory and does not pretend to be self-contained. Details appear in Key's paper [4]. Mathematical background and scientific applications are given in Cohen, Kesten and Newman [3].

Suppose $\{A_j\colon j = 1, 2, \ldots\}$ is a sequence of matrices chosen independently and identically distributed from a finite set of positive $n \times n$ matrices. Suppose $\|A\|$ is any fixed norm of a matrix A. For any positive integer t, let $M_t = A_1 A_2 \ldots A_t$ be the product of the first t matrices from the sequence. Denoting the mathematical expectation or average by the symbol $E(\cdot)$, as is customary in probability theory, the limiting rate of growth of the norm

$$\log \lambda = \lim_{t \to \infty} t^{-1} E(\log \|M_t\|)$$

exists in this example (as well as for many other random sequences $\{A_j\colon j = 1, 2, \ldots\}$). In the degenerate case when all the $A_j = A$ for some fixed positive matrix A, $\log \lambda$ is just the logarithm of the largest eigenvalue of A. So the limiting growth rate denoted by $\log \lambda$ may be thought of as the analog, for products of random matrices, of the logarithm of the largest eigenvalue of a fixed positive matrix.

Let f be a continuous, homogeneous, supermultiplicative function of a positive matrix argument. Here homogeneous means that for $c \geqslant 0$, $f(cA) = cf(A)$. Key [4] proved that for such a function f,

$$\log \lambda = \lim_{t \to \infty} t^{-1} E(\log f(M_t)) \qquad \text{and} \qquad \log \lambda = \lim_{t \to \infty} t^{-1} \log f(M_t)$$

with probability 1, and, moreover, the function f_t defined by $f_t = u^{-1} E \log f(M_u)$, where $u = 2^t$, increases monotonically to $\log \lambda$ with increasing t. Key's result provides a means of bounding $\log \lambda$ from below, namely, by computing f_t for finite values of t.

For sequences of positive matrices, Key cited two nontrivial functions of a positive matrix argument that satisfy the requirements of being continuous, homogeneous and supermultiplicative: the ith diagonal element, and the permanent raised to the power $1/n$. Theorem 1 above expands the set of nontrivial functions that can be used to bound $\log \lambda$ from below, for it implies easily that, for $k = 1, \dots, n$, the functions $\{(A_{[k]})_{ii}\}^{1/k}$ are continuous, homogeneous, and supermultiplicative.

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Assembling r-gons Out of n Given Segments

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In his paper [2], Klamkin proves that if $n \geqslant 3$ positive numbers $a_1, a_2, \dots, a_n$ satisfy

$$\left(\sum_{i=1}^{n} a_i^2\right)^2 > (n-1)\sum_{i=1}^{n} a_i^4, \qquad (1)$$

then any three segments of the lengths a_i, a_j, a_k with $i \neq j \neq k \neq i$ can be assembled into a triangle. Thus the $3 \cdot \binom{n}{3}$ triangle inequalities that are necessary and sufficient for this assembly are replaced by one polynomial inequality of degree 4. Condition (1) is necessary and sufficient for $n = 3$ but only sufficient for $n > 3$.

We present here a rather simple geometric approach to this problem that allows us to replace (1) by a quadratic sufficient inequality (2), extend the result from triangles to r-gons as Klamkin suggested and get a further insight into the problem.

where $u = 2^t$, increases monotonically to $\log \lambda$ with increasing t. Key's result provides a means of bounding $\log \lambda$ from below, namely, by computing f_t for finite values of t.

For sequences of positive matrices, Key cited two nontrivial functions of a positive matrix argument that satisfy the requirements of being continuous, homogeneous and supermultiplicative: the ith diagonal element, and the permanent raised to the power $1/n$. Theorem 1 above expands the set of nontrivial functions that can be used to bound $\log \lambda$ from below, for it implies easily that, for $k = 1, \ldots, n$, the functions $\{(A_{[k]})_{\underline{ii}}\}^{1/k}$ are continuous, homogeneous, and supermultiplicative.

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Assembling *r*-gons Out of *n* Given Segments

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In his paper [2], Klamkin proves that if $n \geqslant 3$ positive numbers $a_1, a_2, \ldots, a_n$ satisfy

$$\left(\sum_{i=1}^{n} a_i^2\right)^2 > (n-1) \sum_{i=1}^{n} a_i^4, \tag{1}$$

then any three segments of the lengths a_i, a_j, a_k with $i \neq j \neq k \neq i$ can be assembled into a triangle. Thus the $3 \cdot \binom{n}{3}$ triangle inequalities that are necessary and sufficient for this assembly are replaced by one polynomial inequality of degree 4. Condition (1) is necessary and sufficient for $n = 3$ but only sufficient for $n > 3$.

We present here a rather simple geometric approach to this problem that allows us to replace (1) by a quadratic sufficient inequality (2), extend the result from triangles to r-gons as Klamkin suggested and get a further insight into the problem.

More precisely, suppose one has n segments of (non-negative) lengths $x_1, x_2, \dots, x_n$ satisfying

$$\left(\sum_{i=1}^{n} x_i\right)^2 \geqslant \left[n - \frac{(r-2)^2}{r}\right] \cdot \sum_{i=1}^{n} x_i^2 \tag{2}$$

for some r, $3 \leqslant r \leqslant n$. Then any r of these n segments in any order can be assembled into an r-gon. (Such a set of n segments will be called *r-realizable.*)

One can easily check that r segments can be assembled into an r-gon in any prescribed order if and only if each of them is not longer than the total of the others.

For a geometric proof of (2), we consider an n-dimensional Euclidean space R^n. To each (ordered) set of n segments, one can assign a point in the nonnegative orthant of R^n whose coordinates are their lengths $x_1, x_2, \dots, x_n$. Denote by K_r the set of points in R^n each of which represents an r-realizable set. Accordingly, a point $(x_1, \dots, x_n) \in K_r$ if and only if any r of its coordinates $x_{i_1}, x_{i_2}, \dots, x_{i_r}$ satisfy

$$x_{i_j} \leqslant \left(\sum_{k=1}^{r} x_{i_k}\right) - x_{i_j}, \qquad j = 1, 2, \dots, r. \tag{3}$$

Thus K_r is the intersection of the $r\binom{n}{r}$ closed half-spaces (3) and hence is a convex polyhedral cone with its vertex at the origin 0.

Obviously the point $U = (1, 1, \dots, 1) \in K_r$. Let us compute the angle α between the vector $\mathbf{U} = \overrightarrow{OU}$ and the hyperplane

$$\sum_{k=1}^{r} x_{i_k} - 2x_{i_j} = 0 \tag{4}$$

bounding the half-space (3) and passing through the origin. A normal $\mathbf{N}$ to that plane is

$$\mathbf{N} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots 0)$$

where all the components are zeros except for those in the positions $i_1, i_2, \dots, i_r$. The latter ones are units except for the position i_j in which -1 is located. One now has

$$r - 2 = \mathbf{U} \cdot \mathbf{N} = \sqrt{n} \cdot \sqrt{r} \sin\alpha; \qquad \sin\alpha = \frac{r-2}{\sqrt{nr}} \tag{5}$$

for any half-space (3).

Obviously, a point $X \neq 0$ belongs to each half-space (3) and thus to K_r if the vector $\mathbf{X} = \overrightarrow{OX}$ forms an angle $\beta \leqslant \alpha$ with $\mathbf{U}$. The latter condition is equivalent to

$$\cos^2\beta \geqslant \cos^2\alpha$$

since the coordinates $x_1, \dots, x_n$ of X are non-negative. It now follows that

$$\left(\frac{\mathbf{X} \cdot \mathbf{U}}{|\mathbf{X}|\,|\mathbf{U}|}\right)^2 = \frac{\left\{\sum_{i=1}^{n} x_i\right\}^2}{n \sum_{i=1}^{n} x_i^2} \geqslant 1 - \frac{(r-2)^2}{nr},$$

which is condition (2).

A few remarks are appropriate here.

1. Conditions (1) and (2) with $r = 3$ are independent in the following sense for $n > 3$. For example, consider the 3-realizable set

$$A = \Big(4, 2, 2, \underbrace{3, 3, \dots 3}_{n-3}\Big). \tag{6}$$

Simple computation shows that relation (2) (with equality in it) holds for the set while (1) fails. On the other hand, (2) fails for the point $B = (0, 1, 1, \dots, 1) \in K_3$ while (1) holds "marginally" (with equality in it instead of $>$). One can also vary B to a point $(\varepsilon, 1, 1, \dots, 1)$ for which (2) fails while (1) holds.

2. FIGURE 1 shows the trihedral angle K_3 for $n = 3$ (intersected by the unit sphere for better visibility). The vectors **u**, **a**, and **b** there are the unit vectors parallel to **U**, **A**, and **B**. The circular cone inscribed in K_3 is determined by (2) while K_3 itself is described by (1). In the dimensions $n > 3$ however, condition (1) determines only a portion of the cone K_3 containing **b** but skipping **a**. This portion is shown symbolically by the dotted line.

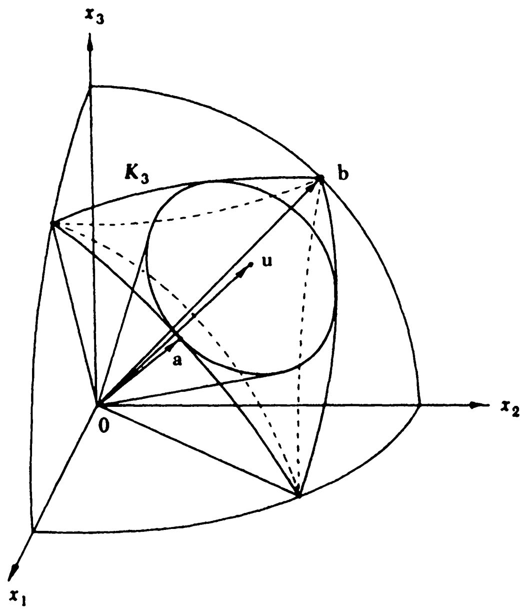

FIGURE 1

3. Klamkin [2] suggests, as an open problem, to find a polynomial inequality like (1) that would be both necessary and sufficient for a set $(a_1, a_2, \dots, a_n)$, $n > 3$, of positive numbers to be 3-realizable. This is hardly possible. Just consider K_3 for $n = 4$ and let ∂K_3 be its boundary. Suppose there is a polynomial $P(x_1, x_2, x_3, x_4)$ such that each point (x_1, x_2, x_3, x_4) in the nonnegative orthant of R^4 belongs to K_3 if and only if $P(x_1, x_2, x_3, x_4) \geqslant 0$. If $P \neq 0$ at a point then $P \neq 0$ and keeps its sign in a neighbourhood of the point. Therefore, the point cannot lie on ∂K_3. Thus $P = 0$ on ∂K_3.

Note that the point $A = (4, 2, 2, 3) \in K_3$ lies on the boundary of the half-space

$$x_1 \leqslant x_2 + x_3 \tag{7}$$

and inside each of the other 11 half-spaces of (3). Therefore, $A \in \partial K_3$ together with a neighbourhood D of A in the 3-plane

$$x_1 = x_2 + x_3. \tag{8}$$

Hence $P = 0$ on D and thus everywhere on the 3-plane (8), due to properties of polynomials.

Take now another point $C = (3, 1, 2, 2)$. It belongs to (8) and also to the boundary

$$x_1 = x_2 + x_4 \tag{9}$$

of the half-space

$$x_1 \leqslant x_2 + x_4 \tag{10}$$

listed in (3). Consider the points of the 3-plane (8) that lie outside of the half-space (10) and are close enough to C to be in the nonnegative orthant. By the definition of P, it should be negative at these points. On the other hand, $P = 0$ there. Thus P does not exist.

Note that our argument in this Remark is applied to a problem slightly different from the original one: our x_i are nonnegative while the original numbers a_i were positive.

A little more elaborate argument involving properties of analytic functions shows that P cannot be analytic either.

4. Having assembled triangles out of n segments, one can try to assemble tetrahedra and, more generally, m-dimensional simplexes. Here n should be $\geqslant \binom{m+1}{2}$. For simplicity, we will only treat tetrahedra. The existence of such tetrahedra depends not only on the lengths of selected 6 segments but also on how these segments are assigned to be the edges of the tetrahedron. The lengths 1, 1, 1, 10, 10, 10 can be assembled as "a 10-tripod on a 1-triangle" but not as "a 1-tripod on a 10-triangle". Let us fix such an assignment. For instance, for an ordered set $S = (a, b, c, d, e, f)$ of six nonnegative numbers, let us assign a to be the edge v_1v_2 between vertices v_1 and v_2 of the tetrahedron, $b = v_1v_3$, $c = v_1v_4$, $d = v_2v_3$, $e = v_2v_4$, and $f = v_3v_4$. The set S will be called *realizable* if the appropriate tetrahedron $v_1v_2v_3v_4$ (possibly degenerate) exists. A necessary and sufficient condition for the set S to be realizable can be found in [**1**], Theorem 1. A simple sufficient condition is that each element of S belongs to $[1/\sqrt{2}, 1]$, [**1**, Theorem 2].

Let $C \subset R^n$ be the set of all points each of which has the following property: Any 6 of its coordinates, taken in their order, form a realizable set. If $(x_1, x_2, \dots, x_n) \in C$, then obviously $(tx_1, tx_2, \dots, tx_n) \in C$ for any $t \geqslant 0$. Thus C is a cone with its vertex at the origin. It would be of interest to study C similarly to K_r and come up with a polynomial inequality such as (1) or (2) sufficient for a point $(x_1, x_2, \dots, x_n)$ to be in C. (Some trivial inequalities of course follow from the fact that the cube $x_i \in [1/\sqrt{2}, 1]$, $i = 1, 2, \dots, n$, belongs to C.) Since $\mathbf{U} \in C$, one might try to inscribe into C a circular cone with the axis parallel to $\mathbf{U}$ as above. There are, however, some difficulties. For instance, C is bounded not by planes but by polynomial surfaces determined by Theorem 1 in [**1**]. The following example shows that C is not convex. Let $n = 6$. Put $P = (1, 5, 9, 4, 8, 4)$ and $Q = (9, 5, 1, 4, 8, 4)$. Both sets are realizable: The points v_1, v_2, v_3 and v_4 lying in this order on a line with $v_1v_2 = 1$, $v_1v_3 = 5$, and $v_1v_4 = 9$ correspond to P. The points v_2, v_3, v_4, and v_1 lying in this order on a line with $v_1v_2 = 9$, $v_1v_3 = 5$, and $v_1v_4 = 1$ correspond to Q. Therefore P and $Q \in C \subset R^6$. However $(P + Q)/2 = (5, 5, 5, 4, 8, 4) \notin C$. Indeed, if $(P + Q)/2 \in C$, then the triangles $v_2v_4v_3$ and $v_2v_4v_1$ determine v_1v_3 as 3 (see FIGURE 2) while it is prescribed to be 5.

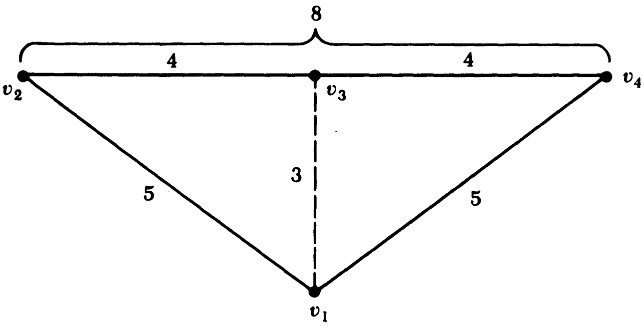

FIGURE 2

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2. Murray S. Klamkin, Simultaneous triangle inequalities, this MAGAZINE 60 (1987), 236–237.

Simultaneous Generalizations of the Theorems of Ceva and Menelaus

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Ceva's and Menelaus's Theorems [1, 2, 3] are very useful for establishing concurrency of lines and collinearity of points, respectively. These theorems can be considered dual to each other [4] and have been generalized from triangles to polygons and spatial figures [3, 5]. Also, one can derive each from the other [6]. Here, we give a generalization for the triangle case that includes both theorems as special cases. Finally, we apply our result to solve two recently posed problems.

We place our triangle in the extended Euclidean plane [3], obtained from the Euclidean plane by adding an ideal line consisting of ideal points such that

(a) each ordinary line of the plane is extended to contain exactly one ideal point;

(b) the members of a family of parallel lines share a common ideal point, distinct families having distinct ideal points.

Let $A_1A_2A_3$ be a triangle in the extended Euclidean plane; b_1, b_2, and b_3 be real numbers; and B_1, B_2, and B_3 be points on the lines A_2A_3, A_3A_1, and A_1A_2, respectively, such that $A_2B_1 = A_2A_3/(1+b_1)$, $A_3B_2 = A_3A_1/(1+b_2)$, and $A_1B_3 = A_1A_2/(1+b_3)$. Note that B_i is ideal if $b_i = -1$. Distance is taken to be signed so that $AB = -BA$. Note that $B_1A_3 = b_1A_2A_3/(1+b_1)$, $B_2A_1 = b_2A_3A_1/(1+b_2)$, and

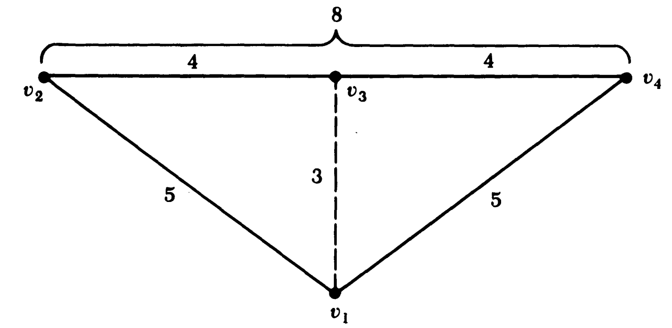

FIGURE 2

REFERENCES

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We place our triangle in the extended Euclidean plane [**3**], obtained from the Euclidean plane by adding an ideal line consisting of ideal points such that

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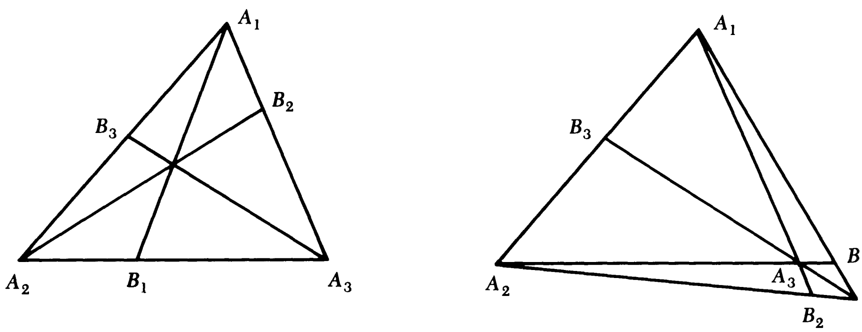
(b) the members of a family of parallel lines share a common ideal point, distinct families having distinct ideal points.

Let $A_1A_2A_3$ be a triangle in the extended Euclidean plane; b_1, b_2, and b_3 be real numbers; and B_1, B_2, and B_3 be points on the lines A_2A_3, A_3A_1, and A_1A_2, respectively, such that $A_2B_1 = A_2A_3/(1+b_1)$, $A_3B_2 = A_3A_1/(1+b_2)$, and $A_1B_3 = A_1A_2/(1+b_3)$. Note that B_i is ideal if $b_i = -1$. Distance is taken to be signed so that $AB = -BA$. Note that $B_1A_3 = b_1A_2A_3/(1+b_1)$, $B_2A_1 = b_2A_3A_1/(1+b_2)$, and

$B_3A_2 = b_3A_1A_2/(1 + b_3)$ so that $A_2B_1/B_1A_3 \cdot A_3B_2/B_2A_1 \cdot A_1B_3/B_3A_2 = 1/b_1b_2b_3$.

Ceva's Theorem (see FIGURE 1) states that A_1B_1, A_2B_2, and A_3B_3 are concurrent if and only if $b_1b_2b_3 = 1$, and Menelaus's Theorem (see FIGURE 2) states that B_1, B_2, and B_3 are collinear if and only if $b_1b_2b_3 = -1$. As given in many texts, Ceva's and Menelaus's Theorems only pertain to the "if" parts. The "only if" parts are given as converse theorems.

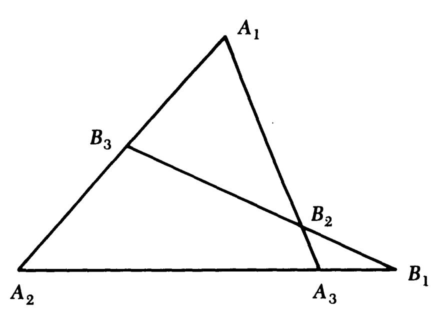
We introduce three additional points C_1, C_2, and C_3, which are on the lines A_2A_3, A_3A_1, and A_1A_2, respectively, and such that $C_1A_3 = A_2A_3/(1 + c_1)$, $C_2A_1 = A_3A_1/(1 + c_2)$, and $C_3A_2 = A_1A_2/(1 + c_3)$, where c_1, c_2, and c_3 are real numbers.



No "exterior" points

Two "exterior" points

FIGURE 1
Ceva's Theorem.



One "exterior" point

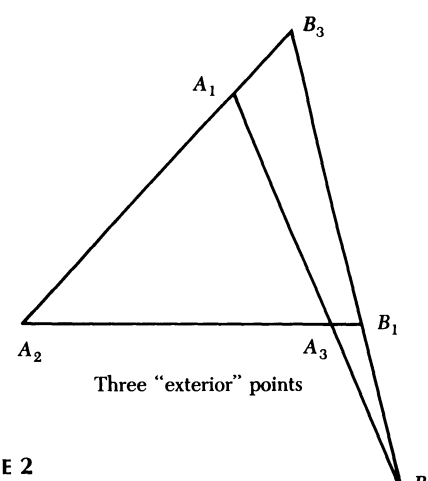


FIGURE 2
Menelaus's Theorem.

Our generalization of both the theorems of Ceva and Menelaus is given by:

THEOREM. *C_1B_2, C_2B_3 and C_3B_1 are concurrent if and only if*

$$b_1b_2b_3 + c_1c_2c_3 + b_1c_1 + b_2c_2 + b_3c_3 = 1. \qquad (*)$$

FIGURE 3 illustrates two of the possible cases. Note that if C_1, C_2, and C_3 coincide with A_2, A_3, and A_1, respectively, then $c_1 = c_2 = c_3 = 0$ so that $(*)$ reduces to $b_1b_2b_3 = 1$. This is Ceva's Theorem. If C_1, C_2, and C_3 coincide with B_1, B_2, and B_3 respectively, then $b_1c_1 = b_2c_2 = b_3c_3 = 1$. Now $(*)$ becomes $b_1b_2b_3 + 1/b_1b_2b_3 = -2$ or $b_1b_2b_3 = -1$. This is Menelaus's Theorem.

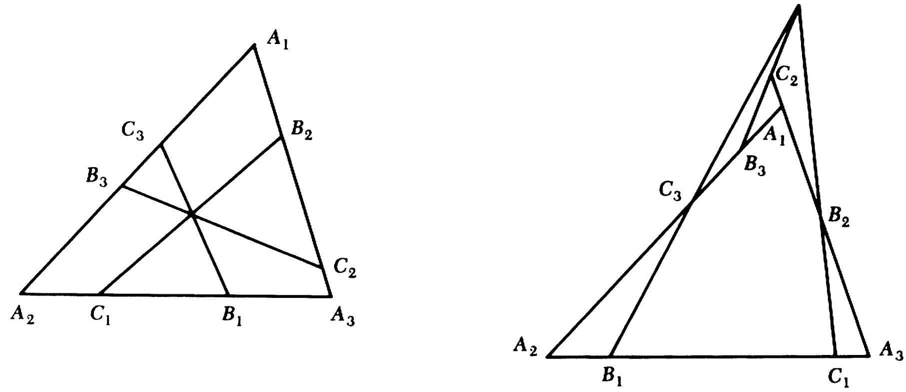


FIGURE 3

To prove our Theorem, we use barycentric coordinates [2] with respect to triangle $A_1A_2A_3$. For an arbitrary point P, its coordinates (p_1, p_2, p_3) are defined by

$$p_1 : p_2 : p_3 = [A_2PA_3] : [A_3PA_1] : [A_1PA_2].$$

Here, $[A_2PA_3]$ is the signed area of triangle A_2PA_3, that is, $[A_2PA_3]$ is positive, zero or negative according to whether P is on the same side of A_2A_3 as A_1, on A_2A_3 or on the opposite side of A_2A_3 to A_1. Similar interpretations apply to $[A_3PA_1]$ and $[A_1PA_2]$. Note that barycentric coordinates are homogeneous, that is, (p_1, p_2, p_3) represents the same point as (kp_1, kp_2, kp_3) for any $k \neq 0$.

We now have the following barycentric coordinates: $B_1(0, b_1, 1)$, $B_2(1, 0, b_2)$, $B_3(b_3, 1, 0)$, $C_1(0, 1, c_1)$, $C_2(c_2, 0, 1)$, and $C_3(1, c_3, 0)$. Let x_1, x_2, and x_3 be the general barycentric coordinate variables. Then the equations of the lines B_1C_3, B_2C_1, and B_3C_2 are, respectively,

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 0 & b_1 & 1 \\ 1 & c_3 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & b_2 \\ 0 & 1 & c_1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x_1 & x_2 & x_3 \\ b_3 & 1 & 0 \\ c_2 & 0 & 1 \end{vmatrix} = 0.$$

These reduce, respectively, to $-c_3x_1 + x_2 - b_1x_3 = 0$, $-b_2x_1 - c_1x_2 + x_3 = 0$, and $x_1 - b_3x_2 - c_2x_3 = 0$. These three lines are concurrent if and only if their equations are linearly dependent. This holds if, and only if,

$$\begin{vmatrix} -c_3 & 1 & -b_1 \\ -b_2 & -c_1 & 1 \\ 1 & -b_3 & -c_2 \end{vmatrix} = 0$$

or $b_1b_2b_3 + c_1c_2c_3 + b_1c_1 + b_2c_2 + b_3c_3 = 1$.

As our first application, we give a short solution to a problem of Guelicher [7] (see FIGURE 4): "In a triangle $P_1P_2P_3$, let p_i be the side opposite vertex P_i, and let s_i be a line parallel to but different from p_i. Suppose that s_i divides P_iP_{i+1} in the signed ratio λ_i, so that if s_i meets p_{i-1} in Q_i, then $\lambda_i = P_iQ_i/Q_iP_{i+1}$, with the subscripts taken modulo 3. Prove that the lines s_1, s_2 and s_3 are concurrent if, and only if, $\lambda_1\lambda_2\lambda_3 - (\lambda_1 + \lambda_2 + \lambda_3) = 2$."

We identify P_1, P_2, P_3, Q_1, Q_2, and Q_3 with A_1, A_2, A_3, C_3, C_1, and C_2, respectively. Then λ_1, λ_2, and λ_3 are equal to c_3, c_1, and c_2, respectively. Take B_1, B_2, and B_3 to be the ideal points on A_2A_3, A_3A_1, and A_1A_2, respectively. Then

$b_1 = b_2 = b_3 = -1$ and $(*)$ simplifies to $c_1c_2c_3 - (c_1 + c_2 + c_3) = 2$, which is equivalent to the equation in the problem.

For applications of this configuration to triangle inequalities and extensions to simplexes, see [8].

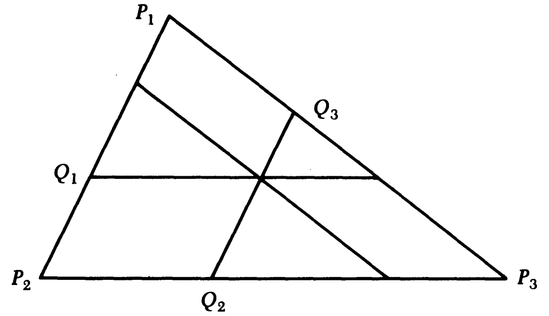


FIGURE 4

As our second application, we prove a concurrency result for perimeter bisectors of a triangle. This was stated without proof by Goggins [9].

Referring to FIGURE 5, let a_1, a_2, and a_3 be the sides A_2A_3, A_3A_1, and A_1A_2 of triangle $A_1A_2A_3$, respectively, with $a_1 > a_2 > a_3$. Let C_1, C_2, and C_3 be the midpoints of A_2A_3, A_3A_1, and A_1A_2, respectively. Let B_1, B_2, and B_3 be points on the perimeter such that C_1B_2, C_2B_3, and C_3B_1 all bisect the perimeter. Then C_1B_2, C_2B_3, and C_3B_1 are concurrent.

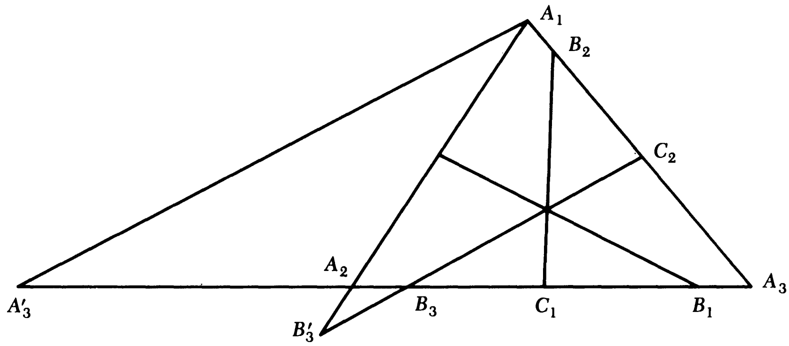


FIGURE 5

To prove this result using our Theorem, we replace B_3 on A_2A_3 by B'_3 on A_1A_2 extended such that B'_3 is collinear with B_3 and C_2. We extend A_3A_2 to A'_3 so that $A'_3B_3 = B_3A_3$. Hence A'_3A_1 is parallel to B_3C_2. Since $A_1A_2 + A_2B_3 = B_3A_3 = A'_3B_3$, $A_1A_2 = A'_3A_2$ and it follows that $A_2B_3 = A_2B'_3$.

Using our standard representation, we have $c_1 = c_2 = c_3 = 1$. Now $A_1B'_3 = A_1A_2/(1 + b_3)$. Hence, $(a_3 + a_1)/2 = a_3/(1 + b_3)$, which simplifies to $b_3 = (a_3 - a_1)/(a_3 + a_1)$. Similarly, $b_1 = (a_1 - a_2)/(a_1 + a_2)$ and

$$b_2 = (a_2 - a_3)/(a_2 + a_3).$$

It is now routine to verify that $(*)$ holds, and the desired concurrency follows.

To conclude this note, we point out that the Dutch artist M. C. Escher (see [10]) considered a configuration that is a special case of our theorem. Referring to FIGURE 6, the side A_2A_3 is subdivided into five equal parts, the side A_3A_1 into four equal

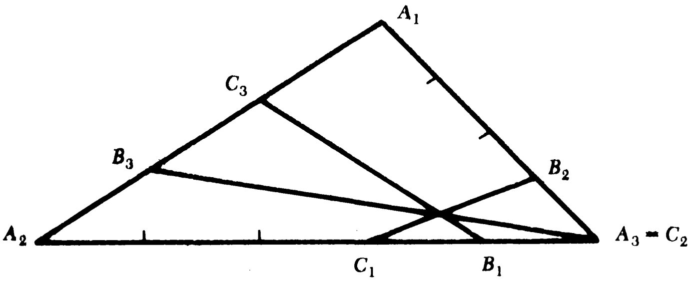


FIGURE 6

parts and the side A_1A_2 into three equal parts. Escher discovered, apparently experimentally, that if B_1 and C_1 are two of the points of division on A_2A_3, possibly the endpoints, and so on, there are twelve cases of concurrency, one of which is shown here.

Our Theorem may be used to verify that he had indeed obtained all the cases. He had five further cases in which three of the points B_1, B_2, B_3, C_1, C_2, and C_3 are internal points of division of one side of $A_1A_2A_3$. While they remind us of our second application, our Theorem does not seem to apply to these cases.

Acknowledgement. The authors are grateful to both referees for valuable suggestions improving the exposition of this note.

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On Some Irrational Series

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The aim of this note is to prove by Euler-type arguments (as in [2], [3], [5]) the following results.

THEOREM 1. *Let* (u_n) *be a bounded sequence of integers with* $u_n \neq 0$ *for infinitely many* n, *and let* (v_n) *be a sequence of positive integers with* $1 < v_n \uparrow \infty$ *and with the property that* $n | v_1 v_2 \cdots v_n$ *for all* n. *Then the series* $\sum_{n=1}^{\infty} u_n / v_1 v_2 \cdots v_n$ *is convergent and its sum is irrational.*

THEOREM 2. *Let* $(u_n), (v_n)$ *be sequences of natural numbers satisfying, for all* n, *the two conditions*:

$$\frac{u_{n+1}}{v_{n+1}} < u_n < v_n, \tag{1}$$

$$n | v_1 v_2 \cdots v_n. \tag{2}$$

Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{u_n}{v_1 v_2 \cdots v_n}$$

is convergent and its sum is irrational.

Proof of Theorem 1. Convergence of the series is immediate by comparison with $\Sigma 1/n(n-1)$ since $n(n-1) | v_1 v_2 \cdots v_n$. Suppose then that

$$\sum_{n=1}^{\infty} \frac{u_n}{v_1 v_2 \cdots v_n} = \frac{M}{N}$$

for integers M and N, $(N > 0)$, and notice that we may suppose this representation to involve an N as large as we please. Then

$$v_1 v_2 \cdots v_N \frac{M}{N} - v_1 v_2 \cdots v_N \sum_{n=1}^{N} \frac{u_n}{v_1 v_2 \cdots v_n} = \sum_{n=N+1}^{\infty} \frac{u_n v_1 v_2 \cdots v_N}{v_1 v_2 \cdots v_n} = R,$$

say. Since $N | v_1 v_2 \cdots v_N$, the left-hand side of this equation is an integer. And, for the right-hand side, writing $B = \max_n |u_n|$, we have the upper estimate:

$$|R| \leqslant \frac{B v_1 v_2 \cdots v_N}{v_1 v_2 \cdots v_{N+1}} \left(1 + \frac{1}{v_{N+2}} + \frac{1}{v_{N+2} v_{N+3}} + \cdots \right)$$

$$\leqslant B \left(\frac{1}{v_{N+1}} + \frac{1}{v_{N+1}^2} + \cdots \right) = \frac{B}{v_{N+1} - 1},$$

which is less than 1 for a sufficiently large (fixed) N. For a lower estimate, let P be the least integer greater than N for which $u_P \neq 0$. Then $|u_P| \geqslant 1$ implies

$$|R| \geqslant v_1v_2 \cdots v_N\left(\frac{1}{v_1v_2 \cdots v_P} - \frac{B}{v_1v_2 \cdots v_{P+1}} - \frac{B}{v_1v_2 \cdots v_{P+2}} - \cdots\right)$$

$$> \frac{v_1v_2 \cdots v_N}{v_1v_2 \cdots v_P}\left[1 - B\left(\frac{1}{v_{P+1}} + \frac{1}{v_{P+1}^2} + \cdots\right)\right] = \frac{v_1v_2 \cdots v_N}{v_1v_2 \cdots v_P}\left(1 - \frac{B}{v_{P+1} - 1}\right).$$

Here one has $v_{P+1} - 1 \geqslant v_{N+1} - 1 > B$, so $|R| > 0$. Therefore $0 < |R| < 1$ with R an integer—a contradiction that completes the proof.

Proof of Theorem 2. First we note that the conditions ensure that $(u_n/v_1v_2 \cdots v_n)$ is monotone decreasing with limit 0, so Leibniz' criterion ensures that the alternating series converges. Suppose now that

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{u_n}{v_1v_2 \cdots v_n} = \frac{M}{N}$$

is rational. Then

$$(-1)^N\left(M\frac{v_1v_2 \cdots v_N}{N} - \sum_{n=1}^{N}(-1)^{N-1}\frac{v_1v_2 \cdots v_Nu_N}{v_1v_2 \cdots v_n}\right)$$

$$= \frac{u_{N+1}}{v_{N+1}} - \frac{u_{N+2}}{v_{N+1}v_{N+2}} + \cdots.$$

By (2), the left-hand side is an integer, while for the sum, R, on the right-hand side we have:

$$R < \frac{u_{N+1}}{v_{N+1}}$$

and

$$R > \frac{u_{N+1}}{v_{N+1}} - \frac{u_{N+2}}{v_{N+1}v_{N+2}}$$

(standard bounds from the theory of alternating series). But (1) implies that $R < 1$, and also that

$$R > \frac{1}{v_{N+1}}\left(u_{N+1} - \frac{u_{N+2}}{v_{N+2}}\right) > 0.$$

Thus R is an integer between 0 and 1—a contradiction as before.

Remark. The proof of Theorem 2 goes through with (1) holding for all sufficiently large n.

For examples of the application of the theorems, consider $v_n = n$. Theorem 1 gives that $\Sigma u_n/n!$ is irrational if (u_n) is bounded with $u_n \neq 0$ for infinitely many n. This implies the irrationality of e, $1/e$ and all of their subseries. (Compare [1].) It also gives a proof that e is not algebraic of degree 2. Indeed, if $ae^2 + be + c = 0$ for some integers a, b, c, then $ae + ce^{-1} = -b$ would be an integer, whilst Theorem 1 with $u_n = a + c(-1)^{n+1}$ shows that $ae + be^{-1}$ is irrational. (Compare [5].) Theorem 2 says that for

$$\frac{u_{n+1}}{(n+1)} < u_n < n$$

we have the irrationality of $\Sigma(-1)^{n-1}u_n/n!$. (With $u_n = 1$, this also implies the irrationality of $1/e$.) For an interesting application, let $u_n = \varphi(n)$, Euler's totient. Then $\varphi(n+1)/(n+1) \leqslant 1 < \varphi(n) < n$ (for $n > 2$), so we obtain the result that $\Sigma(-1)^{n-1}\varphi(n)/n!$ is irrational.

We stress that stronger results than those given in the theorems are provable. For example, Theorem 1 may be deduced from Theorems 1.6 and 1.7 of [4]. The line of argument there is, however, more involved; our aim was to prove something substantially stronger than just "e is irrational" with only slightly more effort than Euler's argument.

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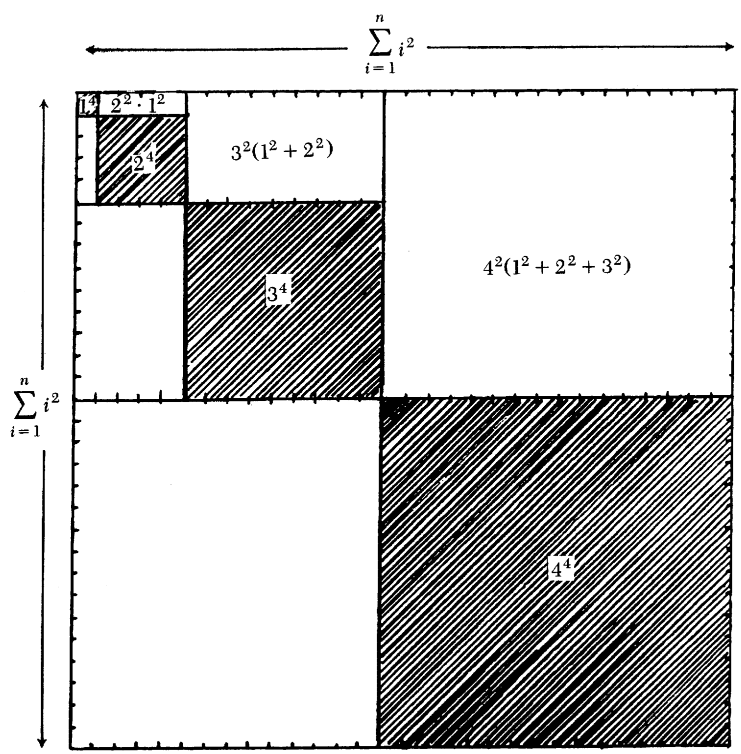
Proof without Words:

$$\sum_{i=1}^{n} i^4 = \left(\sum_{i=1}^{n} i^2\right)^2 - 2\left[\sum_{k=2}^{n}\left(k^2 \sum_{i=1}^{k-1} i^2\right)\right]$$



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that for

$$\frac{u_{n+1}}{(n+1)} < u_n < n$$

we have the irrationality of $\Sigma(-1)^{n-1}u_n/n!$. (With $u_n = 1$, this also implies the irrationality of $1/e$.) For an interesting application, let $u_n = \varphi(n)$, Euler's totient. Then $\varphi(n+1)/(n+1) \leqslant 1 < \varphi(n) < n$ (for $n > 2$), so we obtain the result that $\Sigma(-1)^{n-1}\varphi(n)/n!$ is irrational.

We stress that stronger results than those given in the theorems are provable. For example, Theorem 1 may be deduced from Theorems 1.6 and 1.7 of [4]. The line of argument there is, however, more involved; our aim was to prove something substantially stronger than just "e is irrational" with only slightly more effort than Euler's argument.

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5. G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, Vol. 2, Springer-Verlag, New York, 1976, p. 154 (problems 258, 259), pp. 361–2.

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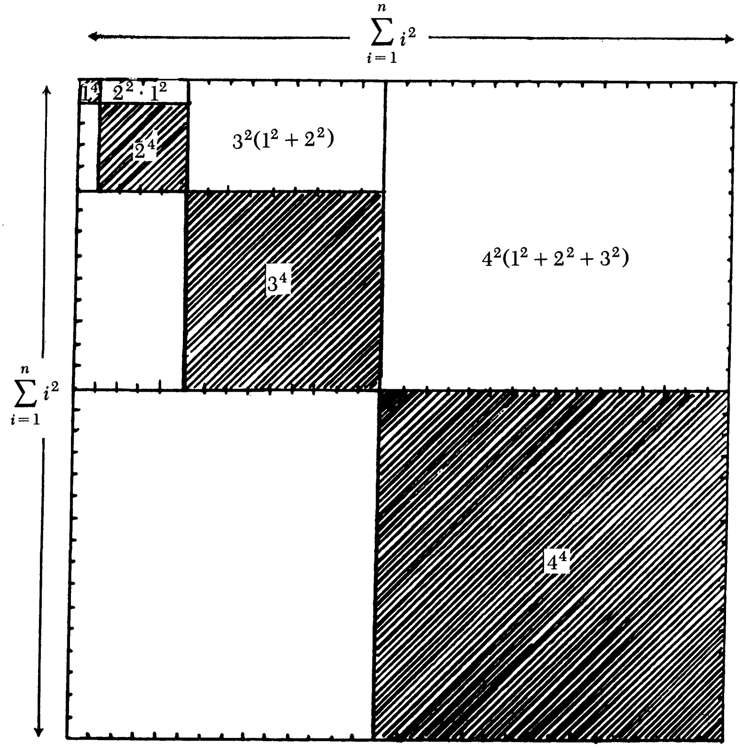

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PROBLEMS

LOREN C. LARSON, *editor*
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Texas Christian University

Proposals

To be considered for publication, solutions should be received by July 1, 1992.

1388. *Proposed by Barry Cipra, Northfield, Minnesota.*

Suppose that n cups are arranged in a circle and that k stones are placed in each cup. Place your hand by one of the cups and carry out the following operation: Pick up all the stones in that cup and, moving clockwise, drop them one at a time into the succeeding cups, leaving your hand by the cup where you dropped the last stone.

a. Prove that by iterating this procedure (always picking up the stones in the cup where you dropped the last stone), you will eventually wind up with all kn stones in the *original* cup.

b*. Note that once the final situation described in part a is reached, one more step will take you back to the original position. Let a_{kn} denote the resulting number of steps. The first few values are given in the following table:

1	4	15	12	75
1	6	21	164	115
1	12	45	164	260
1	8	132	124	3825
1	6	48	1580	1966

Find a formula for a_{kn}.

 We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address.

Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, 1520 St. Olaf Ave., Northfield, MN 55057-1098 or mailed electronically via fax: (507) 663-3549 or e-mail: larson@stolaf.edu.

1389. *Proposed by R. Bruce Richter, Carleton University, Ottawa, Ontario, Canada.*

Evaluate

$$\sum_{j=0}^{n}\binom{2n}{2j}(-3)^{j}.$$

1390. *Proposed by Joseph F. Stephany, Webster Research Center, Webster, New York.*

Prove that no Fibonacci number can be factored into a product of two smaller Fibonacci numbers, each greater than 1.

1391. *Proposed by Howard Morris, Chatsworth, California.*

Show that $1+xe^{x^2/4}\int_0^{x/2}e^{-t^2}\,dt=\sum_{n=0}^{\infty}\frac{n!}{(2n)!}x^{2n}.$

1392. *Proposed by George Andrews, Pennsylvania State University, University Park, Pennsylvania.*

Prove that for any prime p in the interval $(n,4n/3]$, p divides $\sum_{j=0}^{n}\binom{n}{j}^4$.

Quickies

Answers to the Quickies are on page 65.

Q786. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Evaluate the absolute value of the $n\times n$ determinant of the matrix (a_{rs}), where $a_{rs}=\omega^{rs}$, $(r,s=1,2,\ldots,n)$ and ω is a primitive root of $x^n=1$.

Q787. *Proposed by Ioan Sadoveanu, Ellensburg, Washington.*

Find all x for which the sequence $(\tan nx)_{n=1}^{\infty}$ converges.

Solutions

Sums of quotients in Euclid's algorithm **February 1991**

1363. *Proposed by Daniel B. Shapiro, The Ohio State University, Columbus, Ohio.*

Let a,b be positive integers and perform Euclid's algorithm as follows, where $r_0=a$ and $r_1=b$:

$$r_0 = r_1 q_1 + r_2$$
$$r_1 = r_2 q_2 + r_3$$
$$\vdots$$
$$r_{n-2} = r_{n-1} q_{n-1} + r_n$$
$$r_{n-1} = r_n q_n.$$

Then $r_n = (a, b)$ is the greatest common divisor of a and b.

It is easy to show that $\Sigma_{j=1}^n r_j q_j = a + b - (a, b)$, $\Sigma_{j=1}^n r_j^2 q_j = ab$. Show that if $g(x)$ is a polynomial such that the sum

$$\sum_{j=1}^{n} g(r_j) q_j \equiv S_g(a, b)$$

is a polynomial in a, b, and (a, b), then $g(x)$ is a linear combination of x and x^2.

Solution by the proposer.

Since $r_{j-1} = r_j q_j + r_{j+1}$, the following sum telescopes and we have

$$\sum_{j=1}^{n} r_j q_j = \sum_{j=1}^{n} (r_{j-1} - r_{j+1}) = r_0 + r_1 - r_n = a + b - (a, b).$$

If we multiply the jth equation by r_j we have $r_j^2 q_j = r_j r_{j-1} - r_{j+1} r_j$; and again we have a telescoping sum:

$$\sum_{j=1}^{n} r_j^2 q_j = \sum_{j=1}^{n} (r_j r_{j-1} - r_{j+1} r_j) = r_1 r_0 = ab.$$

To prove that $g(x)$ must be a linear combination of x and x^2, suppose more generally that $g(x)$ is a rational function with complex coefficients that is defined at all the positive integers. Suppose that there is some rational function $f(x, y, z)$ with complex coefficients such that $S_g(a, b) = f(a, b, (a, b))$ for all positive integers a, b. Recall the following standard result.

Lemma on Rational Interpolation. Suppose $(r_0, s_0), (r_1, s_1), \ldots, (r_{2d}, s_{2d})$ are in $\mathbf{C} \times \mathbf{C}$ with r_j distinct. Then there exists a unique rational function $k(x) \in \mathbf{C}(x)$ with numerator and denominator polynomials of degree $\leqslant d$ and satisfying $k(r_j) = s_j$ for every $j = 0, 1, \ldots, 2d$.

Let m be the maximum of the degrees of the numerator and denominator of $g(x)$. Let n be the maximum of the y-degrees of the numerator and denominator of $f(x, y, z)$. Let $d = \max\{m + 1, n\}$ and choose a prime number $p > 2d + 1$. Since p and $(2d + 1)!$ are coprime there exists a positive integer a satisfying $a \equiv 1 \pmod{(2d+1)!}$ and $a \equiv 2 \pmod{p}$. Let us compute $S_g(a, k)$ when $1 \leqslant k \leqslant 2d + 1$. Euclid's algorithm in this case has two steps:

$$a = kq + 1 \quad \text{where } q = (a - 1)/k \text{ is an integer,}$$

$$k = 1 \cdot k.$$

Therefore, $S_g(a, k) = g(k)q + g(1)k = g(k)(a - 1)/k + g(1)k$. By hypothesis, this quantity equals $f(a, k, 1)$ for those $2d + 1$ values of k. The Lemma on Rational Interpolation then implies that

$$(a - 1)g(x)/x + g(1)x = f(a, x, 1)$$

where x is an indeterminate. Evaluating at $x = p$ and noting that a, p are coprime, we find that

$$(a-1)g(p)/p + g(1)p = f(a,p,1) = S_g(a,p).$$

Let us now compute $S_g(a,p)$ directly. Euclid's Algorithm for a and p is:

$$a = p \cdot Q + 2 \quad \text{where } Q = (a-2)/p \text{ is an integer,}$$
$$p = 2 \cdot (p-1)/2 + 1$$
$$2 = 1 \cdot 2.$$

Therefore, $S_g(a,p) = (a-2)g(p)/p + g(2)(p-1)/2 + g(1) \cdot 2$. Equating this to the formula derived above and simplifying, we find that

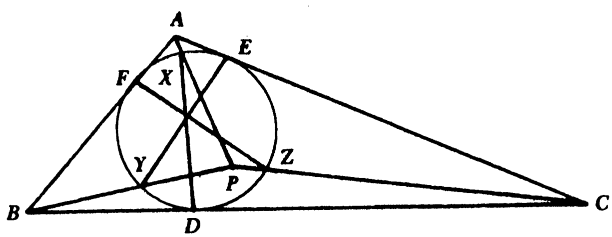
$$g(p)/p + rp + s = 0$$

where $r = g(1) - g(2)/2$ and $s = g(2)/2 - 2g(1)$. This equation is valid for every prime $p > 2d + 1$, and is independent of a. It follows (by the Rational Interpolation Lemma) that $g(x)/x + rx + s = 0$ as a rational function in x. Therefore, $g(x) = -rx^2 - sx$ and we are done.

Points of Rabinowitz **February 1991**

1364. *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

The incircle of triangle ABC touches sides BC, CA, and AB at points D, E, and F, respectively. Let P be any point inside triangle ABC. Line PA meets the incircle at two points; of these, let X be the point that is closer to A. In a similar manner, let Y and Z be the points where PB and PC meet the incircle, respectively. Prove that DX, EY, and FZ are concurrent.



I. *Solution by Francisco Bellot and María Ascensión López, Valladolid, Spain.*

Let $D_1 = \angle XDF$, $D_2 = \angle XDE$; $A_1 = \angle XAF$, $A_2 = \angle XAE$; and name the angles corresponding to the other vertices similarly: E_1, E_2; B_1, B_2; F_1, F_2; C_1, C_2.

We have $D_1 = \angle XFA$, and $D_2 = \angle XEA$. By the Law of Sines,

$$\frac{DX}{\sin DFX} = \frac{FX}{\sin D_1}; \qquad \frac{DX}{\sin DEX} = \frac{EX}{\sin D_2}.$$

But sin DFX = sin DEX because these angles are supplementary, so

$$\frac{FX}{EX} = \frac{\sin D_1}{\sin D_2}. \tag{1}$$

Again, by the Law of Sines, we have

$$\frac{FX}{\sin A_1} = \frac{AX}{\sin D_1}; \qquad \frac{EX}{\sin A_2} = \frac{AX}{\sin D_2},$$

and from this we find that

$$\frac{FX}{EX} = \frac{\sin A_1 \sin D_2}{\sin A_2 \sin D_1}. \tag{2}$$

Therefore, from equations (1) and (2), we have

$$\frac{\sin A_1}{\sin A_2} = \frac{\sin^2 D_1}{\sin^2 D_2}.$$

The same calculations for the other vertices enable us to write

$$\frac{\sin B_1}{\sin B_2} = \frac{\sin^2 E_1}{\sin^2 E_2} \quad \text{and} \quad \frac{\sin C_1}{\sin C_2} = \frac{\sin^2 F_1}{\sin^2 F_2}.$$

Since AX, BY, and CZ are concurrent in P, we know by Ceva's Theorem that

$$\frac{\sin A_1 \sin B_1 \sin C_1}{\sin A_2 \sin B_2 \sin C_2} = 1 = \left(\frac{\sin D_1 \sin E_1 \sin F_1}{\sin D_2 \sin E_2 \sin F_2}\right)^2$$

and therefore, again by Ceva's Theorem, DX, EY, and FZ are concurrent as claimed.

II. *Generalization and solution by René De Vogelaere, University of California, Berkeley, California.*

Preliminary Comments: The following property also holds. If X' is the other intersection of PA and the incircle, and we define similarly Y' and Z', then DX, EY', and FZ' are concurrent in R; moreover, R and the similarly defined points S and T are on the hyperbola passing through A, B, C, P, Q (see below), and the point of Gerrgonne.

The original problem can be rewritten in such a way that, given the incircle, all constructions can be done with the ruler alone, by choosing instead of P, two points Y and Z on the incircle, determining P as the intersection of BY and CZ, Q as the intersection of EY and FZ, and X as the intersection of QD with the incircle distinct from D. Now prove that A, X, and P are collinear.

Alternately, we could start with Q, determine X, Y, Z, and prove the converse property that AX, BY, and CZ are concurrent in P.

In this form the theorem generalizes the projective geometry replacing the incircle by any conic inscribed in the triangle.

Generalization of the converse. (In what follows, the subscript i takes the values 0, 1, and 2, and addition is done modulo 3.) Let A_i be the vertices of a triangle, M a point not on its sides, M_i the intersection of A_iM with the opposite side, let σ be the conic touching the sides of the triangle A_i at the points M_i, let $\overline{M}$ be a point that forms a complete 5-angle with A_i and M, and let $Rabc_i$ be the other intersection of σ with the line $M_i\overline{M}$. Then the lines A_iRabc_i are concurrent in a point Rab. Moreover, if $Rabf_i$ are the other intersections with σ of the lines $RabRabc_i$, then the lines M_iRabc_i, $M_{i+1}Rabf_{i+1}$, and $M_{i-1}Rabf_{i-1}$ are concurrent in Rab_i. Finally, the points $\underline{Rab}$ and Rab_i, which I propose to call the *points of Rabinowitz of* A_i, M, *and* $\overline{M}$, are on a conic κ that passes through the vertices of the triangle, M, and $\overline{M}$.

Proof. Using homogeneous coordinates, with $A_0 = (1,0,0)$, $A_1 = (0,1,0)$, $A_2 = (0,0,1)$, $M = (1,1,1)$, and $\overline{M} = (m_0, m_1, m_2)$, we obtain, for the conic

$$\sigma\colon X_0^2 + X_1^2 + X_2^2 - 2(X_1X_2 + X_2X_0 + X_0X_1) = 0,$$

and for the point,

$$Rabc_0 = \left(4m_0^2, (s_1 - 2m_2)^2, (s_1 - 2m_1)^2\right),$$

with $s_1 = m_0 + m_1 + m_2$.

The line joining A_0 to $Rabc_0$ has homogeneous coordinates

$$rabc_0 = \left[0, (s_1 - 2m_1)^2, -(s_1 - 2m_2)^2\right],$$

and this line with $rabc_1$ and $rabc_2$ have in common

$$Rab = \left((s_1 - 2m_1)^2(s_1 - 2m_2)^2, (s_1 - 2m_2)^2(s_1 - 2m_0)^2, (s_1 - 2m_0)^2(s_1 - 2m_1)^2\right).$$

Moreover, the line joining

$$Rabf_0 = \left(4(m_1 - m_2)^2, (s_1 - 2m_2)^2, (s_1 - 2m_1)^2\right),$$

to M_0 is

$$barg_0 = [m_0, -(m_1 - m_2), m_1 - m_2].$$

Similarly,

$$barg_2 = [-(m_0 - m_1), m_0 - m_1, m_2].$$

Also, the line joining M_1 to $Rabc_1$ is

$$\bar{\iota} M_1 = [m_1, m_1 - m_0, -m_1],$$

and these lines have in common

$$Rab_1 = (m_1 - m_2, m_1, -(m_0 - m_1)).$$

It is easy to verify that Rab and Rab_i are on the conic κ, which has the equation

$$m_0(m_1 - m_2)X_1X_2 + m_1(m_2 - m_0)X_2X_0 + m_2(m_0 - m_1)X_0X_1 = 0.$$

In affine geometry, if M is chosen as the barycenter of the triangle, σ is the inscribed conic of Steiner. In Euclidean geometry, if, moreover, $\overline{M}$ is the orthocenter, then κ is the conic of Kiepert.

Also solved by Jiro Fukuta (Japan), Richard E. Pfiefer, Jyotirmoy Sarkar, Gan Wee Teck (Singapore), Paul Yiu, and the proposer.

Power mean-identric mean inequality **February 1991**

1365. *Proposed by Sidney H. Kung, Jacksonville University, Jacksonville, Florida.*

Prove that for $0 < a < b$,

$$\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$$

I. *Solution by Hans Kappus, Rodersdorf, Switzerland.*

Let us write $b = ax$ with $x > 1$ and take logarithms. Then after rearranging terms the proposed inequality is seen to be equivalent to $f(x) > 0$ for $x > 1$, where the function f is defined by

$$f(x) = x\ln x - (x-1)\left(1 + 2\ln\left(\frac{1+\sqrt{x}}{2}\right)\right); \qquad x \geqslant 1.$$

Now

$$f'(x) = \ln x - 2\ln\left(\frac{1+\sqrt{x}}{2}\right) + \frac{1}{\sqrt{x}} - 1$$

and

$$f''(x) = \frac{\sqrt{x}-1}{2x\sqrt{x}\,(\sqrt{x}+1)}.$$

Obviously, $f(1) = f'(1) = f''(1) = 0$ and $f''(x) > 0$ for $x > 1$. The desired result now follows immediately.

II. *Solution by Michael D. Perlman, University of Washington, Seattle, Washington.* The asserted inequality is equivalent to

$$2\ln\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right) < \frac{b\ln b - a\ln a}{b-a} - 1$$
$$= \frac{1}{b-a}\int_a^b \ln x\,dx$$
$$= \frac{4}{b-a}\int_{\sqrt{a}}^{\sqrt{b}} y\ln y\,dy.$$

Now let $c = \sqrt{a}$, $d = \sqrt{b}$, so this inequality becomes

$$2\ln\left(\frac{c+d}{2}\right) < \frac{4}{d^2-c^2}\int_c^d y\ln y\,dy,$$

which is equivalent to

$$f\left(\frac{c+d}{2}\right) < \frac{1}{d-c}\int_c^d f(y)\,dy,$$

where $f(y) = y\ln y$. But this final inequality is immediate since f is convex on $(0,\infty)$.

III. *Solution and comment by Roger B. Nelsen, Lewis and Clark College, Portland, Oregon.*

To set the notation, suppose that $0 < a < b$ and let $A(k) = A(x, y; k) = ((a^k + b^k)/2)^{1/k}$ (where $k \neq 0$ is a real number) denote the *k-th power mean* of a and b, and $I = I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$ denote the *identric mean* of a and b. The problem asks us to establish that $A(1/2) < I$. In addition, let $A = A(1) = (a+b)/2$ denote the *arithmetic mean* of a and b; $G = G(a,b) = \sqrt{ab}$ denote the *geometric mean* of a and b; and $L = L(a,b) = (b-a)/(\ln b - \ln a)$ denote the *logarithmic mean* of a and b.

It appears that Kenneth B. Stolarsky first introduced the identric mean in [3], where he proved that $G < L < I < A$ (also see [5]). In [4], Stolarsky proved that $A(2/3) < I$, and that the constant $2/3$ is optimal. But it is well known that $A(k)$ is nondecreasing in k [1, 2, 3], and hence $A(1/2) \leqslant A(2/3) < I$, as required.

In a similar vein, Lin [1] has shown that $G < L < A(1/3) < A$, and that the constant $1/3$ in this inequality is also optimal. For a recent survey of these and related results, as well as an extended bibliography, see [2].

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Also solved by Reza Akhlaghi, Larry E. Askins, Christos Athanasiadis (student), Seung-Jin Bang (Korea), Donald Batman, Brian D. Beasley, D. M. Bloom, Paul Bracken (Canada), Barry Brunson, David Callan, Con Amore Problem Group (Denmark), Robert Doucette, David Farnsworth, Jiro Fukuta (Japan), John F. Goehl, Jr., Russell Jay Hendel, Francis M. Henderson, Tommy Konola, Kee-Wai Lau (Hong Kong), Mark G. Leeney (Ireland), Ji-aqi Luo, Phil McCartney, Jean-Marie Monier (France), Robert Patenaude, Bob Prielipp, Heinz-Jürgen Seiffert (Germany), Mohammad Parvez Shaikh (student), Nora Thornber, The University of Indianapolis MAPPS Group, Michael Vowe (Switzerland), Western Maryland College Problems Group, and the proposer.

Series with a fixed sum **February 1991**

1366. *Proposed by Howard Morris, Chatsworth, California.*

Let $\sum_{i=1}^{\infty} a_i$ be a convergent series of positive real numbers, and x an arbitrary positive number. Show there exist uncountably many nondecreasing infinite sequences (b_n) of nonnegative integers, such that $\sum_{i=1}^{\infty} a_i b_i = x$.

Solution by Jerrold W. Grossman, Oakland University, Rochester, Michigan.

We proceed by contradiction, assuming that we have a list of sequences $(b_{1n}), (b_{2n}), \ldots,$ that includes all nondecreasing sequences of nonnegative integers such that $\sum_{i=1}^{\infty} a_i b_{ki} = x$ for each k. Using a diagonalization argument, we will construct another sequence (d_n) that necessarily differs from each sequence (b_{kn}) but nonetheless satisfies the required condition $\sum_{i=1}^{\infty} a_i d_i = x$.

To aid in understanding what is going on we will give this construction geometrically. Imagine a rectangular grid extending infinitely down and to the right, as shown (temporarily ignore the bars):

$$\begin{array}{ccccccc}
\bar{a}_1 & \bar{a}_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
a_1 & \bar{a}_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
a_1 & \bar{a}_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
a_1 & \bar{a}_2 & \bar{a}_3 & \bar{a}_4 & \bar{a}_5 & a_6 & \cdots \\
a_1 & a_2 & a_3 & a_4 & \bar{a}_5 & \bar{a}_6 & \cdots \\
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Each row contains the terms of the original sequence (a_n). Consider all paths through these numbers that start in the upper left corner and proceed only to the right and down, with the proviso that they extend infinitely far to the right. For example, one such path (marked with bars in the display) starts a_1, a_2, a_2, a_2, a_2, a_3, a_4, a_5, a_5, $a_6, \ldots$. Each path cuts out a portion of the grid, namely the portion strictly above the path. This in turn gives us a (possibly divergent) series of the form $\sum_{i=1}^{\infty} a_i b_i$ with (b_n) a nondecreasing sequence of nonnegative integers (this correspondence is clearly one-to-one). In our example, it is $a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 3 + a_4 \cdot 3 + a_5 \cdot 3 + a_6 \cdot 4 + \cdots$. Let $S(P)$ be the sum corresponding to path P.

Now suppose that we are given a countable collection $P_1, P_2, \dots$ of such paths. For our diagonal argument it suffices to construct another path P such that $S(P) = x$ but P differs from each P_k. Set ε equal to x; it will record the remaining error between x and $S(P)$. To begin, have P move at least one step to the right (along the first row) until it extends farther to the right along this row than P_1 does (unless P_1 goes forever along this row, in which case have P move right one step). Let P then turn down, say on the column whose entries are all a_{j_1}. Let $t = \sum_{i=j_1+1}^{\infty} a_i$, and set $c = \lfloor \varepsilon/t \rfloor - 1$. Have P proceed downward c columns (c might be 0, so that P does not actually turn down at this column) and then again turn to the right. Reduce ε by ct, the amount that has now been guaranteed to $S(P)$. Note that by the choice of c, were the path now to continue forever to the right, $\epsilon = x - S(P)$ would be in the interval $[t, 2t)$.

We now carry out this process again: Have P continue to the right until it extends farther along the row that it is currently on than P_2 does (or for one step if it has already passed P_2 or if P_2 goes forever along this row). Let it then turn down, say on the column whose entries are all a_{j_2}. Let $t = \sum_{i=j_2+1}^{\infty} a_i$, and set $c = \lfloor \varepsilon/t \rfloor - 1$. Have P proceed downward c columns and then again turn to the right. Reduce ε by ct, the additional amount contributed to $S(P)$. Repeat again, extending P to the right beyond P_3 in this row (if possible) before turning down, and so on forever. This construction is possible, and P extends infinitely far down, since ε remains strictly positive and the tail sums of $\sum_{i=1}^{\infty} a_i$ approach 0.

Clearly P differs from each P_i, since it extends beyond (or steps before) P_k in some row. Furthermore the construction assures that $S(P) = x$, since the error ε at each stage is a positive number less than twice the sum of the corresponding tail of the original series. Thus we have found a path P—and hence a sequence (b_n)—not in the original list, so our assumption that there were only countably many such sequences is false.

Also solved by Christos Athanasiadis (student), Seung-Jin Bang (Korea), Robert High, Andreas Müller (Switzerland), Stephen Noltie, Adam Reise, Raphael S. Ryger, David Weiland, Western Maryland College Problems Group, and the proposer.

This problem inspired the following result, proposed by the *Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark*: Let $\sum_{i=1}^{\infty} a_i$ be a convergent series of positive real numbers with sum s. There are uncountably many *increasing* infinite sequences (b_n) of positive *real* numbers such that the series $\sum_{i=1}^{\infty} a_i b_i$ is convergent with the same sum s.

Almost Hamiltonian groups — February 1991

1367. *Proposed by John O. Kiltinen, Northern Michigan University, Marquette, Michigan.*

Let G be a group with the property that whenever H and K are subgroups of G, so is $HK = \{hk\colon h \in H \text{ and } k \in K\}$. Does it follow that every subgroup of G is normal? (If every subgroup of G is normal and G is nonabelian, then G is called a Hamiltonian group.)

Solution by the proposer.

No. For p an odd prime, we form a group G of order p^3 determined by generators a and b, which satisfy the relations

$$a^{p^2} = b^p = 1, \qquad ab = ba^{p+1}.$$

The latter relation shows that the subgroup generated by b is not normal. (More concretely, let G be $\mathbf{Z}_p \times \mathbf{Z}_{p^2}$ with operation $(u, v) \cdot (r, s) = (u + r, v + s + vrp)$ and set $a = (0, 1), b = (1, 0)$.)

To show that HK is a subgroup of G for all subgroups H and K, we make use of

$$|HK| = |H| \cdot |K| / |H \cap K|,$$

a result that appears in nearly all introductory algebra texts. It follows that if either H or K has order other than p, then HK is a subgroup. We can see by induction that

$$a^i b^i = b^i a^{j(1+ip)} \quad \text{and} \quad \left(b^i a^j\right)^n = b^{ni} a^{nj + n(n-1)pij/2}.$$

From these, any element of order p can be written in the form $b^i a^{pj}$. and it follows that any two elements of order p commute. Thus, if H and K have order p, their elements commute, and this assures that HK is a subgroup.

Also solved by the Con Amore Problem Group (Denmark), F. J. Flanigan, R. Daniel Hurwitz, and Tricia L. Tripp (student). There was one incorrect solution.

Answers

Solutions to the Quickies on page 57.

A786. The rs-th term of the matrix $(a_{rs})(a_{rs})$ is given by

$$\omega^{r+s} + \omega^{2(r+s)} + \cdots + \omega^{n(r+s)}.$$

The latter is 0 unless $r + s = n$ or $2n$, in which case it is equal to n. Thus,

$$\det(a_{rs})(a_{rs}) = \det\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & n & 0 \\ 0 & 0 & 0 & \cdots & n & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & n & \cdots & 0 & 0 & 0 \\ 0 & n & 0 & \cdots & 0 & 0 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n \end{pmatrix} = \pm n^n$$

and therefore, $|\det(a_{rs})| = n^{n/2}$.

A787. Suppose $\tan nx$ is defined for all n and $\lim_{n \to \infty} \tan nx = L$. Taking the limit of each side of the identity

$$\tan 2nx = \frac{2 \tan nx}{1 - \tan^2 nx},$$

we get

$$L = \frac{2L}{1 - L^2}$$

(if $L = \pm\infty$, we get $\pm\infty = 0$, a contradiction). From this, we find that $L = 0$. Now taking the limit of each side of

$$\tan(n+1)x = \frac{\tan nx + \tan x}{1 - \tan nx \tan x},$$

we find $0 = \tan x$. Therefore, a necessary (and sufficient) condition for convergence is that $x = k\pi$, k an integer.

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Peterson, Ivars, Numbers at random: Number theory supplies a superior random-number generator, *Science News* 140 (9 November 1991) 300–301.

Are you running out of random numbers? Well, help is at hand: Take that old random number generator of yours and supercharge it! To greatly extend its period, you can modify it by adding a "carry bit." George Marsaglia and Arif Zaman (Florida State University) have proved that this new method, as well as a "subtract with borrow" method, will take an ordinary linear-congruential random-number generator with a period of 10^9 or so and "amplify" its randomness to a period of 10^{250} or more. The proofs are number-theoretic, involving primality and factoring.

Kolata, Gina, New method stores data and images better, *New York Times* (12 November 1991) (National Edition) B9.

Wavelets are beginning to hit the public consciousness. This article touts them for their ability to compress data, for example, allowing easier storage and faster comparison with the F.B.I.'s bank of fingerprints. The article describes some of the history of wavelets and highlights the publication by Ingrid Daubechies (AT&T Bell Labs) of a simple collection of wavelets that she proved were the most efficient. A key idea is that wavelets allow for the restoration of an image at varying resolutions.

Dalmédico, Amy Dahan, Sophie Germain, *Scientific American* 265(6) (December 1991) 116–122.

This biographical sketch is accompanied by an account of Germain's scientific career and brief remarks on her work on Fermat's Last Theorem (FLT) and on elasticity. It asserts that Germain made a major step in the case $n = 5$ by showing that one of the variables must be divisible by 5. Dalmédico also claims that Germain proved FLT in the case $n = p - 1$ for a prime p of the form $8k + 7$, quoting Gauss's comments on the proof, which has never been published (Is it even extant? According to Dickson's *History of the Theory of Numbers*, vol. II, p. 732, Germain stated that she could prove this result; but Dickson does not comment further, referring the reader to other sources). This article briefly refers to Germain being the first to define mean curvature, but it does not mention what has become known as Germain's Theorem (Harold M. Edwards, *Fermat's Last Theorem*, 1977, p. 64), which settled Case I of FLT (x, y, z not divisible by n) for all $n < 100$ (Legendre attributed the latter result to her in his 1823 paper, according to Dickson, vol. II, p. 734). Lagrange's equation for Chladni patterns (p. 120) should be in terms of partial derivatives (the *Dictionary of Scientific Biography* entry for Germain gives her equation), and *rentere* on p. 122 should be either *rentier* or *rentiére*.

Peterson, Ivars, Messages in mathematically scrambled waves, *Science News* 140 (3) (20 July 1991) 37–38.

Mathematicians and computer scientists are beginning to investigate cryptology for analog data, such as a phone conversation or a TV signal, without first having to convert the data to digital form. One approach uses an integral operator; a newer approach uses wavelets.

Devaney, Robert L., *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics*, Addison-Wesley, 1990; x + 181 pp, (P). ISBN 0–201–23288–X

Robert Devaney is already the author of a college-level book on dynamical systems (*An Introduction to Chaotic Dynamical Systems*, 2nd ed. Addison-Wesley, 1989). So why this book? "[T]o communicate the vitality of contemporary mathematics to students" at all levels, including high school, and to put the beauty of the field "at their fingertips" Dynamical systems present the opportunity to give mathematics "an experimental component, a laboratory... in which not everything is known." The book requires no more mathematics background than algebra II for the first half and some trigonometry for the second half.

Mathematics Review. 5 issues per year, 46 pp each, A4 size. Philip Allan Publishers Limited, Market Place, Deddington, Oxford OX 4 4SE, England. First subscription $15.95; subsequent subscriptions to same address, $7.50 each.

Aimed at the 16–20 age group, this new mathematics publication "provides teaching resources for ... the A-level curriculum." Because it is devoted solely to mathematics, *Mathematics Review* lacks the emphasis on physical science applications of its American counterpart *Quantum*. But the topics are interesting, the articles are short and lively, the design is very attractive, and the editorial board is distinguished (Christopher Zeeman, Ian Stewart, and colleagues from Warwick). Each issue includes articles (such as determining the date of Ramadan, and the contribution of music to mathematical discovery) plus columns on research news (unsigned but probably written by master expositor Stewart), exam tips, a cartoon series (by Cosgrove), Q & A, and more.

Euler, Leonhard, *Introduction to Analysis of the Infinite, Book II*, transl. John D. Blanton, Springer-Verlag, 1990; xii + 504 pp, $59. ISBN 0–387–97132–7

First English translation of Euler's famous work; this second volume deals with curves and surfaces. (Springer-Verlag and the readers of its books would benefit from higher standards for camera-ready copy, especially for works of great importance, such as this one: The letters and numbers in this book are unacceptably fuzzy and indistinct. For $2–5/page, first-class results could have been rendered by a phototypesetting machine directly from the author's diskette.)

de Mestre, Neville, *The Mathematics of Projectiles in Sport*, Cambridge U Pr, 1990; xi + 175 pp, $22.95 (P). ISBN 0–521–39857–6

Presents a systematic and thorough approach to projectile motion, with examples drawn from the world of sports. Considers motion under gravity, motion with linear and with nonlinear drag, spin effects, numerical solution to devise range tables, perturbation techniques, and corrections due to other effects. The final chapter considers in turn details of the behavior of the projectiles of 20 or so sports. "A basic knowledge of classical dynamics, calculus, vectors, differential equations and their numerical solution is assumed." Exercises are included, without solutions.

Klamkin, Murray S. (ed.), *Problems in Applied Mathematics: Selections from* SIAM Review, SIAM, 1990; xxv + 588 pp. ISBN 0-89871-259-9

Includes about half of the problems (with solutions) from the Problem Section of *SIAM Review* since that journal started in 1959. Here the problems are grouped into 22 categories, by kind of mathematics involved. A person who submits the first-received acceptable solution to any unsolved problem may select any SIAM book as a prize.

Martello, Silvano, and Paolo Toth, *Knapsack Problems: Algorithms and Computer Implementations*, Wiley, 1990; xii + 296 pp, $110. ISBN 0-471-92420-2

Presents the state of the art in algorithms (exact and approximate) for knapsack problems, including change-making, multiple knapsack, bin-packing, and subset-sum problems. Computational experience is described, and a diskette is included with Fortran implementations of the algorithms.

Hubbard, J.H., and B.H. West, *Differential Equations: A Dynamical Systems Approach, Part I*, Springer-Verlag, 1991; xix + 348 pp, $39. ISBN 0-387-97286-2

The age of traditional differential equations books and courses is over. They "focus on techniques leading to solutions. Unfortunately, most differential equations do not admit solutions which can be written in elementary terms. We take the view that a differential equation defines functions; the object of the theory is to understand the behavior of these functions. Our tools include qualitative and numerical methods besides the traditional analytic methods." This text covers qualitative, analytic, and numerical methods; existence and uniqueness; and iteration. The authors introduce some new terminology ("fence," "funnel," "antifunnel") to describe simple phenomena. Companion software is available (for Macintosh). Two subsequent volumes will appear, treating systems of equations and partial differential equations.

Zimmer, Carl, Sandman, *Discover* 12(5) (May 1991) 58-59.

There are more small floods than large ones, more small trees than large, more dim quasars than bright ones, more small avalanches than large ones; and the same quantitative pattern—called *flicker noise*, after the flickering of candle flames—describes them all. Such systems—with many similar interacting components—"don't settle down to a stable configuration but rather push themselves to the verge of instability," called *self-organized criticality*. Physicist Glenn Held (IBM T.J. Watson Research Center) has been experimenting with sandpiles to probe this phenomenon. Held notes that the stock market exhibits flicker noise. Unfortunately, then, like a sandpile or a snowmass, even when it appears calm, "it may be on the verge of a crash." "If the market is comparable to a small sandpile ... it has a self-organized criticality. But what size is the right size for the stock market?"

Kolata, Gina, Computers still can't do beautiful mathematics, *New York Times* (National Edition) (14 July 1991) Section 4, p. 4.

What should a journal do with computer-aided proofs (since a referee can't verify their correctness)? Such proofs have been "festering" ever since the proof of the four-color theorem in 1976, and "many mathematicians won't consider a problem really solved until there is a proof ... that a lone human sitting in a room can go through line by line and understand."

Blakeslee, Sandra, Men's test scores linked to hormone: A cognitive ability in men is influenced by a seasonal cycle, *New York Times* (14 November 1991) (National Edition) A11.

"Men have better spatial ability when tested in spring than in autumn and the variation appears to be linked to seasonal fluctuations in male sex hormones ...." The author of the study speculates that the difference could amount to up to 50 points on the Scholastic Aptitude Test. But spring is when men's testosterone levels tend to be lower, yet older men given "restorative" doses of testosterone improved on tests of spatial reasoning. Moreover, "women whose overall average testosterone level was higher did better on spatial reasoning than other women, no matter where they were in their monthly cycle."

Markoff, John, So who's talking: human or machine?, *New York Times* (5 November 1991) (National Edition) B5, B8; Can machines think? Humans match wits, (9 November 1991) (National Edition) 1, 7. Specter, Michael, Computer program in Queens is mistaken for a human, (16 November 1991) (National Edition) 10; It's alive (but must be plugged in), (18 November 1991) 21.

Alan Turing proposed "Turing's test" to determine if machines can think: Let a person type questions into a terminal; if the person can't tell if the respondent is a person or a computer—but in fact is a computer—then the computer is a "thinking machine." In a competition held at the Boston Computer Museum, with a limited version of the Turing test, the prize was awarded to PC Therapist III. More general versions will be used in future years' contests, with eventually an "open-ended" contest with a prize of $100,000.

Watson, Andrew, Twists, tangles, and topology, *New Scientist* (5 October 1991) 42–46.

Describes recent progress in the mathematical theory of knots in non-technical language. Notes the connection of knots and links with braids, and that Vaughan Jones (now at UC—Berkeley, winner of 1990 Fields Medal) unexpectedly discovered a system to index braid generators and thus label knots and links, using a new polynomial. The article goes on to describe the role of the braid group in statistical physics, that of knot theory in the attempt to classify three-dimensional manifolds , and the connection between knot complements and three-manifolds.

Markoff, John, Scientists devise math tool to break a protective code, *New York Times* (3 October 1991) (National Edition) A16.

Adi Shamir (Weizmann Institute) and Eli Biham (now at the Technion) have notified colleagues that they can break the U.S. Data Encryption Standard (D.E.S.) using a "chosen clear-text attack, "that is, provided they can generate the coded text of any message of their choice.

Ramirez, Anthony, Speeding the calls of fast-talking computers, *New York Times* (13 November 1991) (National Edition) C7.

You may be surprised to learn that the mix of voice and data traffic on U.S. telephone lines is now about half and half and continuing to shift more toward data. This article describes improvements on the traditional technique of packet switching, called *frame relay* (bigger but variable-size packets, less error-checking) and *cell relay* (bigger but fixed-size packets, allowing standardization of equipment), which are more suitable for fiber-optic networks.

NEWS AND LETTERS

LETTERS TO THE EDITOR

Dear Editor:

Norman Schaumberger's concise article "A Coordinate Approach to the AM-GM Inequality," which appeared in the October 1991 issue, gives a method that may be used to establish the rest of the usual mean inequality statement: $A \geq G \geq H$. H is the harmonic mean. Consider the curve $y = exp(Hx)$. It is concave upward with the tangent line $y = Hex$ at the point $(1/H, e)$. Substituting $x = 1/a_i$ in $exp(Hx) \geq Hex$ successively and multiplying gives $G \geq H$.

Gary Miller
Fredericton, N.B.
Canada

Dear Editor:

Many proofs of the AM-GM Inequality have appeared, so I guess you probably wouldn't mind seeing one more. The well-known Bernoulli's inequality [1] states that if $x > 0$,

$$x^r - 1 \geq r(x - 1) \qquad (*)$$

for any $r > 1$. The equality holds if and only if $x = 1$. Let $a_1, a_2, \ldots, a_n$ be positive numbers with arithmetic mean A_n and geometric mean G_n. If we take $r = k > 1$ ($k = 2, 3, \ldots, n$) and put $x = \sqrt[k-1]{a_k/A_k}$ in (*) we get

$$\frac{\sqrt[k-1]{a_k(a_1 + a_2 + \ldots + a_{k-1})}}{k-1} \leq \sqrt[k-1]{A^k_k},$$

or

$$\sqrt[k-1]{a_k A_{k-1}} \leq \sqrt[k-1]{A^k_k}.$$

Hence

$$a_k A^{k-1}_{k-1} \leq A^k_k;$$

and by induction, we have

$$a_n a_{n-1} \ldots a_2 A_1 \leq A^n_n.$$

Therefore,

$$G_n = \sqrt[n]{a_n a_{n-1} \ldots a_2 a_1} \leq A_n.$$

My point is: By properly choosing x and placing it in (*), we shall be able to obtain a host of other important inequalities (for example, letting $x = n \cdot a_i/\Sigma a_i$ ($i = 1, 2, \ldots, n$) in (*) yields the power mean inequality $(\Sigma a_i/n)^k \leq (1/n)\, \Sigma a^k_i$). Thus it seems that Bernoulli's inequality is a prime source of a class of classical inequalities.

Reference:

1. G.H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, New York, 1964, p. 42.

Sidney H. Kung
Jacksonville University
Jacksonville, FL

Dear Editor:

Taking normal care of a pocket calculator usually includes not sitting on it. But I'd like to share with *MAGAZINE* readers a folk remedy for ailing calculators. My Sharp EL-512 II went on the fritz, just about all the digits being formed with missing legs. Months later, still defunct, I carelessly put it in my back pocket, intending to take it to a store in hopes of getting a replacement. But after a day in this environment, the calculator came back to life, all but one digit being fully restored. Obviously, the only thing to do was return it to my seat pocket for more treatment. It worked, and the calculator has performed flawlessly ever since!

David Callan
University of Wisconsin
Madison, WI

ICME-7 will be held in Quebec, Canada, from 16 through 23 August 1992. For more information on registration, contact:

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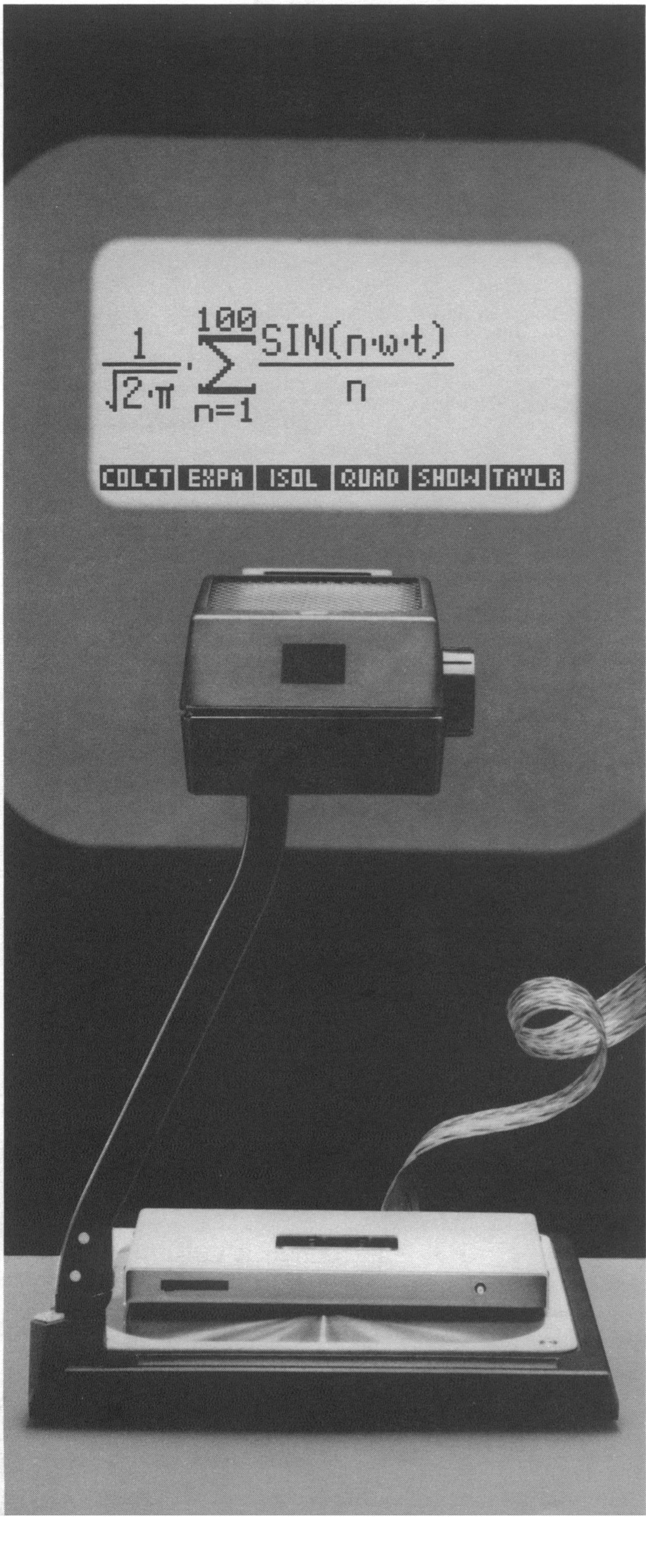

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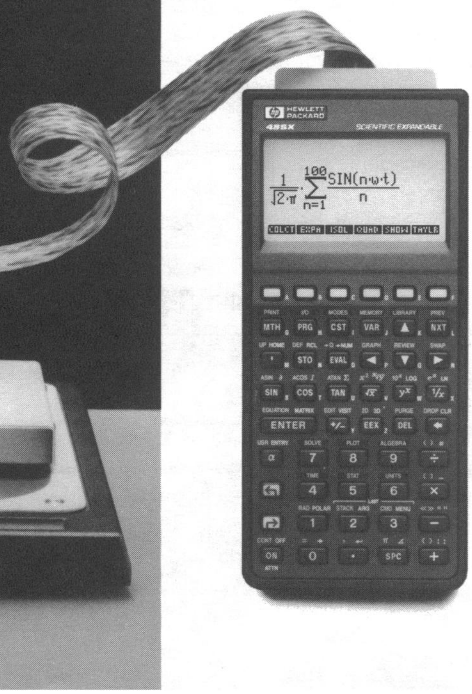

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